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International Journal of
Information
and
Management
Sciences

International Journal of
Information and Management Sciences

37 (2026), 1-20. DOI:10.6186/IJIMS.202603_37(1).0001

Strategic Behavior Analysis of Fluid Queue with Incomplete Fault and Repair Delay

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Keywords

Fluid Queue, Incomplete Fault, Repair Delay, Benefit Function, Balking Strategy

Abstract

This paper analyzes a fluid model with incomplete fault and repair delay from an economic perspective under fully and almost observable cases. The system alternates among repair, working, and incomplete fault states, with exponentially distributed durations. The arriving fluid calculates its own net benefit based on the observed system information to determine whether to enter the system. The individual equilibrium balking threshold is obtained with conditional expectation theorem. The steady probability distribution of fluid level is derived by using the eigenvector method of linear differential equations with constant coefficients, and the optimal balking strategy of social benefit per unit time are discussed. Finally, some numerical examples are presented to show the influence of system parameters on social benefits per unit time and the SOA algorithm is applied to seek the optimal threshold and optimal social benefit. The results provide the reference for the optimal design and management of the system.

1. Introduction

Queuing theory plays an important role in the modeling and analysis of stochastic service systems. In recent years, the study of customer strategic behavior in queuing system from the perspective of economics has become a hot topic in stochastic operation research and management science. Naor (1969) first proposed an equilibrium analysis of customer behavior in M/M/1 queue system under the observable case.

The service system may experience faults or even stop running at any time during the service process, and the types of faults are diverse. Such as a computer, the service efficiency may be reduced due to hardware aging and stutter, or the service may be stopped directly due to complete damage. Economou and Kanta (2008) considered customer equilibrium strategy of a queuing system with complete failure and immediate repair in the observable case. Li et al.

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(2014) extended the research results of literature Economou and Kanta (2008) and analyzed the customer balking strategy under the unobservable case. Xu and Xu (2018) analyzed the balking strategy of customer and the average social benefits of the M/M/1 queuing model with incomplete failure and delayed maintenance characteristics under the fully observable and the fully unobservable cases. Yang et al. (2014) conducted an equilibrium analysis of discrete queuing system with fault characteristics.

With the development of computer technology, the interarrival time of customers and service time in the queuing system are getting shorter and shorter, and the discrete customers are getting closer and closer to the continuous fluid. The traditional discrete time queuing system show great limitations and inconvenience in modeling analysis and the fluid queue becomes more and more important. Li and Ma (2008) studied a class of fluid models with perturbation. Vijayalakshmi and Thangaraj (2013) discussed fluid models driven by M/M/1 queues with catastrophe. Vijayashree and Anjuka (2016) studied the fluid queuing model driven by M/M/1 queue with working vacation and described the dynamic changing of buffer capacity with the parameters by numerical examples. Xu et al. (2018) studied a fluid model driven by M/M/1 queueing system with working vacation and negative customer. Xu et al. (2017) gave the empty probability of the buffer and the average fluid level in the fluid queue modulated by the PH/M/1 queue.

Parallel to the economic analysis of the queuing model, Maglaras (2006) studied revenue management for a multiclass single-server queue via a fluid model. Economou and Manou (2016) analyzed the observable fluid model with an alternating service process, and derive the individual equilibrium threshold and the social optimal strategy. Furthermore, Wang and Xu (2018) established a fluid vacation queuing model with start-up policy, and deduced the individual equilibrium balking strategy and the social optimization strategy per unit time. Wang and Xu (2021) studied the strategic behavior of fluid models with working vacation. Xu and Wang (2022) studied a unobservable fluid model with alternating on-off states. Considering the heterogeneity of customers, Logothetis et al. (2023) conducted a comparative analysis of customer decision behaviors in an on-off fluid queuing system with reneging and without reneging, focusing on issues such as equilibrium throughput and social welfare. In recent years, Xu et al. (2021) studied the observable fluid queuing systems with fault and disaster, where disaster arrival causes the system to fail and stop service, forcing all customers to leave. The Nash equilibrium balking strategy of customers is obtained, and the research results provide inspiration for individuals and decision makers to achieve optimal benefit.

In addition, the fluid model has been successfully applied to the modeling and simulation of different types of service systems. Considering node dormancy and node failure problem in the data transmission process of mobile Internet, Zhang and Xu (2024) studied the equilibrium strategy of fluid queuing model with fault maintenance and working vacation in the fully observable case. Given that unrestricted access to a highway network under overload may lead to throughput loss and other problems, Helton et al. (2021) analyzed a fluid model of a traffic network with information feedback and onramp controls. Based on the a considering the working mechanism of the electronic inventory system, when a fault warning occurs during normal operation, the system will stop receiving new orders and process existing tasks at a low rate prior to maintenance. During the maintenance period, the system will not handle any tasks but will accept new orders so that it can start working immediately after maintenance. Based on the above analysis, this paper introduces the semi-fault and repair delay into the fluid queuing model, and studies the individual equilibrium strategy and the average social optimal strategy in the fully observable and almost observable cases, respectively.

The rest of this paper is organized as follows: Section 2 constructs a fluid queuing model with incomplete fault and repair delay. Sections 3 and 4 study the individual equilibrium strategy and the average social benefit per unit time of the fluid under the fully observable case and the almost observable case, respectively. Section 5 uses numerical analysis to demonstrate the influence of key parameters on social benefits, and SOA algorithm is designed to find the social optimal joining thresholds and the maximum social profit. Section 6 summarizes the content, and puts forward the prospect of the future research work.

2. Model description

Assume that the fluid flow into the buffer at a rate of λ. The buffer alternatively stays in repair state, working state and incomplete fault state, and the duration of the three states follows the exponential distribution of parameters $\theta_0,\theta_1,\theta_2$, respectively. During the repairment period, the fluid can flow into the buffer but the system stops serving, no fluid flows out. In the working state, the outflow rate of the fluid is μ_1; In the incomplete fault state, the buffer serves at a lower rate μ_0 and does not allow fluid to enter. Let $X(t)$ be the fluid level in the buffer at the time of t, $I(t)=0,1,2$ indicates that the buffer is in the repair period, working period and incomplete fault period at time t, respectively. Then the net input rate structure of the buffer can be described as follows

$$\frac{dX(t)}{dt}=\begin{cases}\lambda, & I(t)=0,X(t)\geq 0\\ \lambda-\mu_1, & I(t)=1,X(t)>0\\ -\mu_0, & I(t)=2,X(t)>0\end{cases}$$

Based on its own net income, the fluid will decide whether to enter the buffer. When the net income is not negative, the fluid is willing to enter the buffer, otherwise it tends to balk. The decision of the fluid whether to flow into the buffer is not reversible, that is, the entering fluid cannot renege, and the balked fluid cannot enter the buffer again. Assuming that all fluids are indistinguishable and try to maximize their own benefits based on the behavior of other fluids, this case can be viewed as a game between fluids. Assume that the fluid flowing out of the buffer after service gets revenue R, but there is a loss of C per unit time during the sojourn period of the fluid in the system. In the following, the customer behavior will be analyzed in the fully observable and almost observable cases, respectively.

3. Equilibrium analysis in the fully observable case

3.1 Equilibrium balking strategy

In the fully observable case, the fluid level and buffer status in the system are known for the arriving fluid. The system information $(X(t),I(t))=(x,i)$ represents the fluid level in the buffer is x and the buffer is in the state i at time t. Denote the utility function of the fluid entering the system when the system information is (x,i) by

$$B_i(x)=R-CE\left(S_i(x)\right),i=0,1,2$$

where $E\left(S_i(x)\right)$ is the average sojourn time of the fluid entering the system when the system information is (x,i).

Remark 1 In the fully observable case, in order to ensure that the fluid that arrives when the system information is (0,1) must enter the system, assume that $R > \frac{C}{\mu_1}$.

Theorem 1 In the observable case, when the system information is (x,i), the average sojourn time of the fluid entering the system is as follows

$$E(S_0(x)) = \frac{1}{\theta_0} + E(S_1(x)),$$

$$E(S_1(x)) = \frac{\left[\frac{1}{\mu_1}-\frac{1}{\mu_0}\left(1+\frac{\theta_2}{\theta_0}\right)\right]}{\left(\frac{\theta_1}{\mu_1}+\frac{\theta_2}{\mu_0}\right)^2}\frac{\theta_1}{\mu_1}\left[1-e^{-\left(\frac{\theta_1}{\mu_1}+\frac{\theta_2}{\mu_0}\right)x}\right]+\frac{\left[\theta_2+\theta_1\left(1+\frac{\theta_2}{\theta_0}\right)\right]}{\mu_1\mu_0\left(\frac{\theta_1}{\mu_1}+\frac{\theta_2}{\mu_0}\right)}x,$$

$$E(S_2(x)) = \frac{\left[\frac{1}{\mu_1}-\frac{1}{\mu_0}\left(1+\frac{\theta_2}{\theta_0}\right)\right]}{\left(\frac{\theta_1}{\mu_1}+\frac{\theta_2}{\mu_0}\right)^2}\frac{\theta_2}{\mu_0}\left[1-e^{-\left(\frac{\theta_1}{\mu_1}+\frac{\theta_2}{\mu_0}\right)x}\right]+\frac{\left[\theta_2+\theta_1\left(1+\frac{\theta_2}{\theta_0}\right)\right]}{\mu_1\mu_0\left(\frac{\theta_1}{\mu_1}+\frac{\theta_2}{\mu_0}\right)}x$$

According to theorem 1, we know that $E\left(S_i(x)\right)$ monotonically increases with respect to x in $[0,+\infty)$, so $B_i(x)$ monotonically decreases with respect to x in $[0,+\infty)$.

Theorem 2 In the fully observable case, denote the individual equilibrium threshold of the fluid by $x_e(i), i=0,1,2$, where $x_e(i)$ is the only solution to the equation $B_i(x) = R - CE\left(S_i(x)\right) = 0$. The individual equilibrium strategy can be described as follows: when the arriving fluid finds the system information is (x,i), and the fluid is more willing to enter the buffer if $x < x_e(i)$; the fluid balks if $x > x_e(i)$; the fluid remains neutral if $x = x_e(i)$.

3.2 Socially optimal strategy

Because fluids pay attention to their own benefits and game each other, the individual equilibrium thresholds do not necessarily lead to optimal social benefits. In the fully observable case, assume that the social optimal threshold per unit time is $x_*(i), i=0,1,2$, the steady-state probability distribution of the fluid level of the buffer at state $i\ (i=0,1,2)$ is denoted by

$$F_i(x) = \lim_{t\to+\infty} P\left(X(t)\le x, I(t)=i\right), x\ge 0; i=0,1,2.$$

Theorem 3 In the fully observable case, when the fluid obeys the threshold strategy $\left(x_*(0), x_*(1), x_*(2)\right)$, the steady-state probability distribution of the fluid level in the buffer at state $i(i=0,1,2)$ can be expressed as follows

(I) When $\lambda > \mu_1$, then

$$F_0(x) = \begin{cases} 0, & x\le 0 \\ c_{00}+c_{01}e^{h_1 x}+c_{02}e^{h_2 x}, & 0\le x < x_*(0) \\ \dfrac{\theta_1\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x\ge x_*(0) \end{cases}$$

$$F_1(x)=\begin{cases} 0, & x\le 0 \\ c_{00}\dfrac{\theta_0}{\theta_1}+c_{01}\dfrac{\theta_0+\lambda h_1}{\theta_1}e^{h_1x}+c_{02}\dfrac{\theta_0+\lambda h_2}{\theta_1}e^{h_2x}, & 0<x\le x_*(1) \\ \dfrac{\theta_0\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x\ge x_*(1) \end{cases}$$

$$F_2(x)=\begin{cases} 0, & x\le 0 \\ c_{00}\dfrac{\theta_0}{\theta_2}+c_{01}\dfrac{\theta_0}{\theta_2-\mu_0h_1}e^{h_1x}+c_{02}\dfrac{\theta_0}{\theta_2-\mu_0h_2}, & 0<x\le x_*(1) \\ \dfrac{\theta_0\theta_1}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x\ge x_*(1) \end{cases}$$

The probability mass at the discontinuity points of the distribution function are given by

$$P_0(x_*(0))=\frac{\theta_1\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}-c_{00}-c_{01}e^{h_1x_*(0)}-c_{02}e^{h_2x_*(0)}$$

$$P_1(0)=c_{00}\frac{\theta_0}{\theta_1}+c_{01}\frac{\theta+\lambda h_1}{\theta_1}+c_{02}\frac{\theta_0+\lambda h_2}{\theta_1}, \qquad P_2(0)=c_{00}\frac{\theta_0}{\theta_2}+c_{01}\frac{\theta_0}{\theta_2-\mu_0h_1}+c_{02}\frac{\theta_0}{\theta_2-\mu_0h_2}$$

where

$$\begin{aligned} c_{01}=&\frac{\theta_0\theta_1\theta_2}{\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2} \\ &\times\frac{(\theta_2-\mu_0h_1)B_1}{\left[(\lambda h_1+\theta_0)e^{h_1x_*(1)}-\theta_0\right](\theta_2-\mu_0h_1)B_1+\left[(\lambda h_2+\theta_0)e^{h_1x_*(1)}-\theta_0e^{(h_1-h_2)x_*(1)}\right](\theta_2-\mu_0h_2)A_1} \end{aligned}$$

$$\begin{aligned} c_{02}=&\frac{\theta_0\theta_1\theta_2}{\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2} \\ &\times\frac{(\theta_2-\mu_0h_2)A_1}{\left[(\lambda h_2+\theta_0)e^{h_2x_*(1)}-\theta_0\right](\theta_2-\mu_0h_2)A_1+\left[(\lambda h_1+\theta_0)e^{h_2x_*(1)}-\theta_0e^{(h_2-h_1)x_*(1)}\right](\theta_2-\mu_0h_1)B_1} \end{aligned}$$

$$c_{00}=-(c_{01}+c_{02}), \qquad A_1=\lambda\mu_0h_1^2+\theta_0\mu_0h_1-\lambda\theta_2h_1, \quad B_1=\lambda\mu_0h_2^2+\theta_0\mu_0h_2-\lambda\theta_2h_2$$

$$h_1=\frac{\dfrac{\theta_2}{\mu_0}-\dfrac{\theta_0}{\lambda}-\dfrac{\theta_1}{\lambda-\mu_1}+\sqrt{\Delta}}{2}, \quad h_2=\frac{\dfrac{\theta_2}{\mu_0}-\dfrac{\theta_0}{\lambda}-\dfrac{\theta_1}{\lambda-\mu_1}-\sqrt{\Delta}}{2},$$

$$\Delta=\left(\frac{\theta_1}{\lambda-\mu_1}+\frac{\theta_0}{\lambda}-\frac{\theta_2}{\mu_0}\right)^2-4\left(\frac{\theta_0\theta_1}{\lambda(\lambda-\mu_1)}-\frac{\theta_0\theta_2}{\lambda\mu_0}-\frac{\theta_1\theta_2}{\mu_0(\lambda-\mu_1)}\right)$$

(II) When $\lambda = \mu_1$, then

$$F_0(x) = \begin{cases} 0, & x \le 0 \\ \dfrac{\theta_0\theta_1\theta_2\mu_0\left(e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x}-1\right)}{\left(\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2\right)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*(0)}-\theta_0\mu_0\right)}, & 0 \le x < x_*(0) \\ \dfrac{\theta_1\theta_2}{\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2}, & x \ge x_*(0) \end{cases}$$

$$F_1(x) = \begin{cases} 0, & x < x_*(0) \\ \dfrac{\theta_0\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x \quad x_*(0) \end{cases}$$

$$F_2(x) = \begin{cases} 0, & x \le 0 \\ \dfrac{\theta_0\theta_1\left(\lambda\theta_2 e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x}-\mu_0\theta_0\right)}{\left(\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2\right)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*(0)}-\theta_0\mu_0\right)}, & 0 < x \le x_*\left(0\right) \\ \dfrac{\theta_0\theta_1}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x \ge x_*\left(0\right) \end{cases}$$

The probability mass at the discontinuity points of the distribution function are given by

$$P_0(x_*(0)) = \frac{\theta_1\theta_2}{\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2} - \frac{\theta_0\theta_1\theta_2\mu_0\left(e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*(0)}-1\right)}{\left(\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2\right)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*(0)}-\theta_0\mu_0\right)}$$

$$P_1\left(x_*\left(0\right)\right) = \frac{\theta_0\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2} \qquad P_2(0) = \frac{\theta_0\theta_1\left(\lambda\theta_2-\mu_0\theta_0\right)}{\left(\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2\right)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*\left(0\right)}-\theta_0\mu_0\right)}$$

(III) When $\lambda < \mu_1$, then

$$F_0(x) = \begin{cases} 0, & x \le 0 \\ c_{10} + c_{11}e^{h_1 x} + c_{12}e^{h_2 x}, & 0 \le x < x_*(0) \\ \dfrac{\theta_1\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2}, & x \ge x_*(0) \end{cases}$$

$$F_1(x) = \begin{cases} 0, & x \le 0 \\ c_{10}\dfrac{\theta_0}{\theta_1} + c_{11}\dfrac{\theta_0 + \lambda h_1}{\theta_1}e^{h_1 x} + c_{12}\dfrac{\theta_0 + \lambda h_2}{\theta_1}e^{h_2 x}, & 0 < x \le x_*(0) \\ \dfrac{\theta_0\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2}, & x \ge x_*(0) \end{cases}$$

$$F_2(x) = \begin{cases} 0, & x \le 0 \\ c_{10}\dfrac{\theta_0}{\theta_2} + c_{11}\dfrac{\theta_0}{\theta_2 - \mu_0 h_1}e^{h_1 x} + c_{12}\dfrac{\theta_0}{\theta_2 - \mu_0 h_2}e^{h_2 x}, & 0 < x \le x_*(0) \\ \dfrac{\theta_0\theta_1}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2}, & x \ge x_*(0) \end{cases}$$

The probability mass at the discontinuity points of the distribution function are given by

$$P_0(x_*(0)) = \frac{\theta_1\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2} - c_{10} - c_{11}e^{h_1 x_*(0)} - c_{12}e^{h_2 x_*(0)},$$

$$P_1(0) = c_{10}\frac{\theta_0}{\theta_1} + c_{11}\frac{\theta_0 + \lambda h_1}{\theta_1} + c_{12}\frac{\theta_0 + \lambda h_2}{\theta_1}$$

where

$$c_{10} = -(c_{11} + c_{12})$$

$$c_{11} = \frac{\theta_0\theta_1\theta_2}{\theta_0\theta_1 + \theta_0\theta_2 + \theta_1\theta_2} \times \frac{(\theta_2 - \mu_0 h_1)B_1}{\left[(\lambda h_1 + \theta_0)e^{h_1 x_*(0)} - \theta_0\right](\theta_2 - \mu_0 h_1)B_1 + \left[(\lambda h_2 + \theta_0)e^{h_1 x_*(0)} - \theta_0 e^{(h_1 - h_2)x_*(0)}\right](\theta_2 - \mu_0 h_2)A_1}$$

$$c_{12} = \frac{\theta_0\theta_1\theta_2}{\theta_0\theta_1 + \theta_0\theta_2 + \theta_1\theta_2} \times \frac{(\theta_2 - \mu_0 h_2)A_1}{\left[(\lambda h_2 + \theta_0)e^{h_2 x_*(0)} - \theta_0\right](\theta_2 - \mu_0 h_2)A_1 + \left[(\lambda h_1 + \theta_0)e^{h_2 x_*(0)} - \theta_0 e^{(h_2 - h_1)x_*(0)}\right](\theta_2 - \mu_0 h_1)B_1}$$

Proof. The stationary distribution of fluid level in the buffer are discussed in the following three cases.

(I) When $\lambda > \mu_1$, the system of ordinary differential equations is established according to the assumptions of the model

$$\begin{cases} \lambda \dfrac{dF_0(x)}{dx} = -\theta_0 F_0(x) + \theta_1 F_1(x) \\ (\lambda - \mu_1)\dfrac{dF_1(x)}{dx} = -\theta_1 F_1(x) + \theta_2 F_2(x) \\ -\mu_0 \dfrac{dF_2(x)}{dx} = -\theta_2 F_2(x) + \theta_0 F_0(x) \end{cases} \tag{3.8}$$

with the boundary conditions

$$F_0(0) = 0, \quad F_1(x_*(1)) = \frac{\theta_0\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2}, \quad F_2(x_*(1)) = \frac{\theta_0\theta_1}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2} \tag{3.9}$$

Denoted by $\dfrac{dF_i(x)}{dx} = F_i'(x), i = 0,1,2$, then the system of differential equations (3.8) can be written in matrix form

$$\begin{bmatrix} F_0'(x) \\ F_1'(x) \\ F_2'(x) \end{bmatrix} = \begin{bmatrix} -\dfrac{\theta_0}{\lambda} & \dfrac{\theta_1}{\lambda} & 0 \\ 0 & -\dfrac{\theta_1}{\lambda - \mu_1} & \dfrac{\theta_2}{\lambda - \mu_1} \\ -\dfrac{\theta_0}{\mu_0} & 0 & \dfrac{\theta_2}{\mu_0} \end{bmatrix} \begin{bmatrix} F_0(x) \\ F_1(x) \\ F_2(x) \end{bmatrix} = A \begin{bmatrix} F_0(x) \\ F_1(x) \\ F_2(x) \end{bmatrix} \tag{3.10}$$

To solve the equation (10), compute the eigenvalue of the matrix A, then we have

$$h_0 = 0, \quad h_1 = \frac{\dfrac{\theta_2}{\mu_0} - \dfrac{\theta_0}{\lambda} - \dfrac{\theta_1}{\lambda - \mu_1} + \sqrt{\Delta}}{2}, \quad h_2 = \frac{\dfrac{\theta_2}{\mu_0} - \dfrac{\theta_0}{\lambda} - \dfrac{\theta_1}{\lambda - \mu_1} - \sqrt{\Delta}}{2}$$

where

$$\Delta = \left(\frac{\theta_1}{\lambda - \mu_1} + \frac{\theta_0}{\lambda} - \frac{\theta_2}{\mu_0} \right)^2 - 4\left(\frac{\theta_0\theta_1}{\lambda(\lambda - \mu_1)} - \frac{\theta_0\theta_2}{\lambda\mu_0} - \frac{\theta_1\theta_2}{\mu_0(\lambda - \mu_1)} \right)$$

Solving the eigenvector corresponding to the eigenvalue $h_i, i = 0,1,2$ of the matrix A, respectively

$$V_0 = \begin{bmatrix} 1 \\ \dfrac{\theta_0}{\theta_1} \\ \dfrac{\theta_0}{\theta_2} \end{bmatrix}, V_1 = \begin{bmatrix} 1 \\ \dfrac{\lambda h_1 + \theta_0}{\theta_1} \\ \dfrac{\theta_0}{\theta_2 - \mu_0 h_1} \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ \dfrac{\lambda h_2 + \theta_0}{\theta_1} \\ \dfrac{\theta_0}{\theta_2 - \mu_0 h_2} \end{bmatrix}$$

By using the eigenvector method of linear differential equations with constant coefficients, the general solution expression of equation (3.10) is obtained

$$\begin{bmatrix} F_0(x) \\ F_1(x) \\ F_2(x) \end{bmatrix} = c_{00} V_0 + c_{01} V_1 e^{h_1 x} + c_{02} V_2 e^{h_2 x}$$

Combined with boundary conditions (3.9), coefficient of c_{00}, c_{01}, c_{02} can be obtained, so the results in case (I) are proved.

Similarly, the results in cases (II) and (III) can be proved.

In the fully observable case, assuming all fluid follows social threshold $(x_*(0), x_*(1), x_*(2))$, then the social benefit per unit time is

$$B(x_*(0), x_*(1), x_*(2)) = \lambda_e R - CE(X) \tag{3.14}$$

where λ_e is the effective arrival rate of the fluid, $E(X)$ is the average fluid level in steady state.

Theorem 4 In the observable case, the social benefit per unit time can be expressed as

$$\begin{aligned} & B(x_*(0), x_*(1), x_*(2)) \\ & = \begin{cases} \lambda\left(P_1(0) + \int_0^{x_*(0)} f_0(x)\,dx + \int_0^{x_*(1)} f_1(x)\,dx\right)R - C[x_*(0)P_0(x_*(0)) + x_*(1)P_1(x_*(1)) & \\ \quad + \int_0^{x_*(0)} x f_0(x)\,dx + \int_0^{x_*(1)} x(f_1(x) + f_2(x))\,dx], & \lambda > \mu_1 \\ \lambda\left(\dfrac{\theta_0\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2} + \int_0^{x_*(0)} f_0(x)\,dx\right)R - C[x_*(0)P_0(x_*(0)) + \int_0^{x_*(0)} x(f_0(x) + f_2(x))\,dx & \\ \quad + x_*(0)\dfrac{\theta_0\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2}], & \lambda = \mu_1 \\ \lambda\left(\dfrac{\theta_0\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2} + \int_0^{x_*(0)} f_0(x)\,dx\right)R - C[\int_0^{x_*(0)} x(f_0(x) + f_1(x) + f_2(x))\,dx & \\ \quad + x_*(0)P_0(x_*(0))], & \lambda < \mu_1 \end{cases} \end{aligned}$$

Proof. The social benefit per unit time is discussed in the following three cases.

(I) When $\lambda > \mu_1$, the fluid level of the buffer changes within the interval $[0, x_*(1)]$. When the fluid level is equal to $x_*(0)$, no fluid is willing to enter if the buffer is in state 0. When the fluid level reaches $x_*(1)$ and buffer is in state 1, the fluid enters the buffer with probability $\frac{\mu_1}{\lambda}$. When the fluid level is between $(0,x_*(0))$, the fluid flows in with the probability of $f_0(x)$ if the buffer is in state 0, the fluid flows in with the probability of $f_1(x)$ if the buffer is in state 1. When the fluid level is between $(x_*(0),x_*(1))$, no fluid is selected to enter if the buffer is in state 0, the fluid enters with the probability of $f_1(x)$ if the buffer is in state 1.

From the results of case (I) in theorem 3, we obtain the probability density functions in the three states

$$f_0(x)=F_0'(x)=c_{01}h_1e^{h_1x}+c_{02}h_2e^{h_2x},0<x<x_*(0)$$

$$f_1(x)=F_1'(x)=c_{01}h_1\frac{\theta_0+\lambda h_1}{\theta_1}e^{h_1x}+c_{02}h_2\frac{\theta_0+\lambda h_2}{\theta_1}e^{h_2x},0<x<x_*(1)$$

$$f_2(x)=F_2'(x)=c_{01}h_1\frac{\theta_0}{\theta_2-\mu_0h_1}e^{h_1x}+c_{02}h_2\frac{\theta_0}{\theta_2-\mu_0h_2}e^{h_2x},0<x<x_*(1)$$

Combined with the above analysis, the effective arrival rate of the fluid is

$$\lambda_e=\lambda\left(P_1(0)+\int_0^{x_*(0)}f_0(x)dx+\int_0^{x_*(1)}f_1(x)dx\right) \tag{3.15}$$

The average fluid level of the system is

$$E(X)=x_*(0)P_0(x_*(0))+x_*(1)P_1(x_*(1))+\int_0^{x_*(0)}xf_0(x)dx+\int_0^{x_*(1)}x\left(f_1(x)+f_2(x)\right)dx \tag{3.16}$$

(II) When $\lambda=\mu_1$, similarly, based on the results of case (II) in theorem 3, we obtain the probability density functions in the three states

$$f_0(x)=\frac{\theta_0\theta_1\theta_2\mu_0\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x}}{(\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*(0)}-\theta_0\mu_0\right)},0<x<x_*(0)$$

$$f_2(x)=\frac{\theta_0\theta_1\theta_2\lambda\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x}}{(\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*(0)}-\theta_0\mu_0\right)},0<x<x_*(0)$$

Thus, the effective arrival rate of the fluid is

$$\lambda_e=\lambda\left(\frac{\theta_0\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}+\int_0^{x_*(0)}f_0(x)dx\right) \tag{3.17}$$

The average fluid level of the system is

$$E(X)=x_*(0)P_0\left(x_*(0)\right)+\int_0^{x_*(0)} x\left(f_0(x)+f_2(x)\right)dx+x_*(0)\frac{\theta_0\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2} \tag{3.18}$$

(III) When $\lambda < \mu_1$, similarly, the effective arrival rate of the fluid and the average fluid level of the buffer can be derived, respectively

$$\lambda_e=\lambda\left(\frac{\theta_0\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}+\int_0^{x_*(0)} f_0(x)dx\right) \tag{3.19}$$

$$E(X)=x_*(0)P_0\left(x_*(0)\right)+\int_0^{x_*(0)} x\left(f_0(x)+f_1(x)+f_2(x)\right)dx \tag{3.20}$$

where

$$f_0(x)=c_{11}h_1e^{h_1x}+c_{12}h_2e^{h_2x}, 0<x<x_*(0)$$

$$f_1(x)=c_{11}h_1\frac{\theta_0+\lambda h_1}{\theta_1}e^{h_1x}+c_{12}h_2\frac{\theta_0+\lambda h_2}{\theta_1}e^{h_2x}, 0<x<x_*(0)$$

$$f_2(x)=c_{11}h_1\frac{\theta_0}{\theta_2-\mu_0h_1}e^{h_1x}+c_{12}h_2\frac{\theta_0}{\theta_2-\mu_0h_2}e^{h_2x}, 0<x<x_*(0)$$

Based on the above analysis, the analytic expression of the average social benefit per unit time can be obtained in three cases by substituting equations (3.15) and (3.16), (3.17) and (3.18), (3.19) and (3.20) into equation (3.13), respectively.

Finally, solving the optimization problem of $\arg\max\{B(x_*(0),x_*(1),x_*(2))\}$ then we get the maximum point of the average social benefit per unit time, that is, the social optimal threshold $x_*(i), i=0,1,2$.

4. Equilibrium analysis in the almost observable case

In the almost observable case, the arriving fluid observes the fluid level $X(t)$ in the buffer and cannot ascertain the buffer state $I(t)$. When the arriving fluid observes the fluid level x in the buffer, the individual benefit function of the fluid upon entering the system is denoted as $B(x)=R-CE\left(S(x)\right)$, where $S(x)$ represents the average sojourn time of the fluid in the buffer when the fluid level in the buffer is observed to be x when it arrives at the system. The fluid tends to enter the buffer when $B(x)>0$. Denote the individual equilibrium balking threshold by x_e, that is, $B(x_e)=0$. When the fluid reaches the buffer, the fluid selects to enter if the fluid level x is found to be lower than the threshold x_e; the fluid balks if the fluid level x is higher than the threshold x_e.

Theorem 5 In the almost observable cases, when the fluid reaches the system and the fluid level in the buffer is observed to be x, the average sojourn time of the fluid in

the buffer $E(S(x))$ is as follows.

(I) When $\lambda > \mu_1$,

$$E(S(x))=\begin{cases}\dfrac{\theta_2}{\theta_1+\theta_2}E\left(S_1(x)\right)+\dfrac{\theta_1}{\theta_1+\theta_2}E\left(S_2(x)\right), & x=0\\ \dfrac{f_0(x)E\left(S_0(x)\right)}{f_0(x)+f_1(x)+f_2(x)}+\dfrac{f_1(x)E\left(S_1(x)\right)}{f_0(x)+f_1(x)+f_2(x)}+\dfrac{f_2(x)E\left(S_2(x)\right)}{f_0(x)+f_1(x)+f_2(x)}, & 0<x<x_e\\ E\left(S_0(x)\right), & x=x_e\end{cases} \tag{4.1}$$

(II) When $\lambda=\mu_1$,

$$E(S(x))=\begin{cases}E\left(S_2(x)\right), & x=0\\ \dfrac{f_0(x)}{f_0(x)+f_2(x)}E\left(S_0(x)\right)+\dfrac{f_2(x)}{f_0(x)+f_2(x)}E\left(S_2(x)\right), & 0<x<x_e\\ \dfrac{\theta_1}{\theta_0+\theta_1}E\left(S_0(x)\right)+\dfrac{\theta_0}{\theta_0+\theta_1}E\left(S_1(x)\right), & x=x_e\end{cases} \tag{4.2}$$

(III) When $\lambda<\mu_1$, the results are the same as (4.1).

Furthermore, according to the expression of individual benefit function and theorem 5, solving the solution to the equation $B(x)=R-CE(x)=0$, the individual equilibrium balking strategy under the almost observable cases can be obtained.

In the almost observable case, to optimize the social benefit per unit time, it is assumed that all fluids adhere to the social threshold strategy x_*. The steady-state probability distribution of the fluid level when the buffer is in state $i\ (i=0,1,2)$ is

$$F_i(x)=\lim_{t\to+\infty}P\left(X(t)\le x, I(t)=i\right), x\ge 0; i=0,1,2$$

Similar to the proof of theorem 3, the following results can be obtained.

Theorem 6 In the almost observable case, when all fluids follow the social threshold strategy x_*, the steady-state probability distribution of the fluid level for the buffer at state $i(i=0,1,2)$ can be expressed as follows

(I) When $\lambda>\mu_1$,

$$F_0(x)=\begin{cases}0, & x\le 0\\ d_{00}+d_{01}e^{h_1x}+d_{02}e^{h_2x}, & 0<x\le x_*\\ \dfrac{\theta_1\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x>x_*\end{cases}$$

$$F_1(x)=\begin{cases}0, & x\le 0\\ d_{00}\dfrac{\theta_0}{\theta_1}+d_{01}\dfrac{\theta_0+\lambda h_1}{\theta_1}e^{h_1x}+d_{02}\dfrac{\theta_0+\lambda h_2}{\theta_1}e^{h_2x}, & 0<x\le x_*\\ \dfrac{\theta_0\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x>x_*\end{cases}$$

$$F_2(x)=\begin{cases} 0, & x\le 0\\ d_{00}\dfrac{\theta_0}{\theta_2}+d_{01}\dfrac{\theta_0}{\theta_2-\mu_0h_1}e^{h_1x}+d_{02}\dfrac{\theta_0}{\theta_2-\mu_0h_2}, & 0<x\le x_*\\ \dfrac{\theta_0\theta_1}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x>x_* \end{cases}$$

The probability mass at the discontinuity points of the distribution function are given by

$$P_0(x_*)=\frac{\theta_1\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}-d_{00}-d_{01}e^{h_1x_*}-d_{02}e^{h_2x_*}$$

$$P_1(0)=d_{00}\frac{\theta_0}{\theta_1}+d_{01}\frac{\theta+\lambda h_1}{\theta_1}+d_{02}\frac{\theta_0+\lambda h_2}{\theta_1}, \qquad P_2(0)=d_{00}\frac{\theta_0}{\theta_2}+d_{01}\frac{\theta_0}{\theta_2-\mu_0h_1}+d_{02}\frac{\theta_0}{\theta_2-\mu_0h_2}$$

where

$$d_{00}=-\left(d_{01}+d_{02}\right)$$

$$d_{01}=\frac{\theta_0\theta_1\theta_2}{\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2}\times\frac{\left(\theta_2-\mu_0h_1\right)B_1}{\left[\left(\lambda h_1+\theta_0\right)e^{h_1x_*}-\theta_0\right]\left(\theta_2-\mu_0h_1\right)B_1+\left[\left(\lambda h_2+\theta_0\right)e^{h_1x_*}-\theta_0e^{\left(h_1-h_2\right)x_*}\right]\left(\theta_2-\mu_0h_2\right)A_1}$$

$$d_{02}=\frac{\theta_0\theta_1\theta_2}{\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2}\times\frac{\left(\theta_2-\mu_0h_2\right)A_1}{\left[\left(\lambda h_2+\theta_0\right)e^{h_2x_*}-\theta_0\right]\left(\theta_2-\mu_0h_2\right)A_1+\left[\left(\lambda h_1+\theta_0\right)e^{h_2x_*}-\theta_0e^{\left(h_2-h_1\right)x_*}\right]\left(\theta_2-\mu_0h_1\right)B_1}$$

(II) When $\lambda=\mu_1$,

$$F_0(x)=\begin{cases} 0, & x\le 0\\ \dfrac{\theta_0\theta_1\theta_2\mu_0\left(e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x}-1\right)}{\left(\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2\right)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*}-\theta_0\mu_0\right)}, & 0<x\le x_*\\ \dfrac{\theta_1\theta_2}{\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2}, & x>x_* \end{cases}$$

$$F_1(x)=\begin{cases} 0, & x<x_*\\ \dfrac{\theta_0\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x\quad x_* \end{cases}$$

$$F_2(x)=\begin{cases}0, & x\le 0\\ \dfrac{\theta_0\theta_1\left(\lambda\theta_2 e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x}-\mu_0\theta_0\right)}{\left(\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2\right)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*}-\theta_0\mu_0\right)}, & 0<x\le x_*\\ \dfrac{\theta_0\theta_1}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x>x_*\end{cases}$$

The probability mass at the discontinuity points of the distribution function are given by

$$P_0(x_*)=\frac{\theta_1\theta_2}{\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2}-\frac{\theta_0\theta_1\theta_2\mu_0\left(e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*}-1\right)}{\left(\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2\right)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*}-\theta_0\mu_0\right)}$$

$$P_1\left(x_*\right)=\frac{\theta_0\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, \qquad P_2(0)=\frac{\theta_0\theta_1\left(\lambda\theta_2-\mu_0\theta_0\right)}{\left(\theta_0\theta_1+\theta_0\theta_2+\theta_1\theta_2\right)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0}-\frac{\theta_0}{\lambda}\right)x_*}-\theta_0\mu_0\right)}$$

(III) When $\lambda<\mu_1$,

$$F_0(x)=\begin{cases}0, & x\le 0\\ d_{10}+d_{11}e^{h_1x}+d_{12}e^{h_2x}, & 0<x\le x_*\\ \dfrac{\theta_1\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x>x_*\end{cases}$$

$$F_1(x)=\begin{cases}0, & x\le 0\\ d_{10}\dfrac{\theta_0}{\theta_1}+d_{11}\dfrac{\theta_0+\lambda h_1}{\theta_1}e^{h_1x}+d_{12}\dfrac{\theta_0+\lambda h_2}{\theta_1}e^{h_2x}, & 0<x\le x_*\\ \dfrac{\theta_0\theta_2}{\theta_0\theta_1+\theta_1\theta_2+\theta_0\theta_2}, & x>x_*\end{cases}$$

$$F_2(x) = \begin{cases} 0, & x \le 0 \\ d_{10}\dfrac{\theta_0}{\theta_2} + d_{11}\dfrac{\theta_0}{\theta_2 - \mu_0 h_1} e^{h_1 x} + d_{12}\dfrac{\theta_0}{\theta_2 - \mu_0 h_2} e^{h_2 x}, & 0 < x \le x_* \\ \dfrac{\theta_0\theta_1}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2}, & x > x_* \end{cases}$$

The probability mass at the discontinuity points of the distribution function are given by

$$P_0(x_*) = \frac{\theta_1\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2} - d_{10} - d_{11}e^{h_1 x_*} - d_{12}e^{h_2 x_*}, \quad P_1(0) = d_{10}\frac{\theta_0}{\theta_1} + d_{11}\frac{\theta_0 + \lambda h_1}{\theta_1} + d_{12}\frac{\theta_0 + \lambda h_2}{\theta_1}$$

where

$$d_{10} = -(d_{11} + d_{12})$$

$$d_{11} = \frac{\theta_0\theta_1\theta_2}{\theta_0\theta_1 + \theta_0\theta_2 + \theta_1\theta_2} \frac{(\theta_2 - \mu_0 h_1)B_1}{\left[(\lambda h_1 + \theta_0)e^{h_1 x_*} - \theta_0\right](\theta_2 - \mu_0 h_1)B_1 + \left[(\lambda h_2 + \theta_0)e^{h_1 x_*} - \theta_0 e^{(h_1 - h_2)x_*}\right](\theta_2 - \mu_0 h_2)A_1}$$

$$d_{12} = \frac{\theta_0\theta_1\theta_2}{\theta_0\theta_1 + \theta_0\theta_2 + \theta_1\theta_2} \frac{(\theta_2 - \mu_0 h_2)A_1}{\left[(\lambda h_2 + \theta_0)e^{h_2 x_*} - \theta_0\right](\theta_2 - \mu_0 h_2)A_1 + \left[(\lambda h_1 + \theta_0)e^{h_2 x_*} - \theta_0 e^{(h_2 - h_1)x_*}\right](\theta_2 - \mu_0 h_1)B_1}$$

In the almost observable case, assume that all fluids follow the social threshold strategy x_*, then x_* is the maximum point of the average social benefit per unit time. The average social benefit function per unit time is denoted by $B(x) = \lambda_e R - CE(X)$, where λ_e is the effective arrival rate and $E(X)$ is the average fluid level.

According to theorem 6, the effective arrival rate of the fluid λ_e and the average fluid level $E(X)$ respectively are as follows

(I) When $\lambda > \mu_1$,

$$\lambda_e = \lambda\left(P_1(x_*)\frac{\mu_1}{\lambda} + \int_0^{x_*}\left(f_0(x) + f_1(x)\right)dx\right), \quad E(X) = x_* P_0(x_*) + x_* P_1(x_*) + \int_0^{x_*} x\left(f_0(x) + f_1(x) + f_2(x)\right)dx$$

where

$$f_0(x) = d_{01}h_1 e^{h_1 x} + d_{02}h_2 e^{h_2 x}, 0 < x < x_*$$

$$f_1(x) = d_{01}h_1\frac{\theta_0 + \lambda h_1}{\theta_1}e^{h_1 x} + d_{02}h_2\frac{\theta_0 + \lambda h_2}{\theta_1}e^{h_2 x}, 0 < x < x_*$$

$$f_2(x) = d_{01}h_1\frac{\theta_0}{\theta_2 - \mu_0 h_1}e^{h_1 x} + d_{02}h_2\frac{\theta_0}{\theta_2 - \mu_0 h_2}e^{h_2 x}, 0 < x < x_*$$

(II) When $\lambda = \mu_1$,

$$\lambda_e = \lambda\left(\frac{\theta_0\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2} + \int_0^{x_*} f_0(x)dx\right)$$

$$E(X) = x_* P_0(x_*) + \int_0^{x_*} x\left(f_0(x) + f_2(x)\right)dx + x_*\frac{\theta_0\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2}$$

where

$$f_0(x) = \frac{\theta_0\theta_1\theta_2\mu\left(\frac{\theta_2}{\mu_0} - \frac{\theta_0}{\lambda}\right)e^{\left(\frac{\theta_2}{\mu_0} - \frac{\theta_0}{\lambda}\right)x}}{\left(\theta_0\theta_1 + \theta_0\theta_2 + \theta_1\theta_2\right)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0} - \frac{\theta_0}{\lambda}\right)x_*} - \theta_0\mu_0\right)}, 0 < x < x_*$$

$$f_2(x) = \frac{\theta_0\theta_1\theta_2\lambda\left(\frac{\theta_2}{\mu_0} - \frac{\theta_0}{\lambda}\right)e^{\left(\frac{\theta_2}{\mu_0} - \frac{\theta_0}{\lambda}\right)x}}{\left(\theta_0\theta_1 + \theta_0\theta_2 + \theta_1\theta_2\right)\left(\theta_2\lambda e^{\left(\frac{\theta_2}{\mu_0} - \frac{\theta_0}{\lambda}\right)x_*} - \theta_0\mu_0\right)}, 0 < x < x_*$$

(III) When $\lambda < \mu_1$,

$$\lambda_e = \lambda\left(\frac{\theta_0\theta_2}{\theta_0\theta_1 + \theta_1\theta_2 + \theta_0\theta_2} + \int_0^{x_*} f_0(x)dx\right), \quad E(X) = x_* P_0(x_*) + \int_0^{x_*} x\left(f_0(x) + f_1(x) + f_2(x)\right)dx$$

where

$$f_0(x) = d_{11}h_1e^{h_1x} + d_{12}h_2e^{h_2x}, 0 < x < x_*$$

$$f_1(x) = d_{11}h_1\frac{\theta_0 + \lambda h_1}{\theta_1}e^{h_1x} + d_{12}h_2\frac{\theta_0 + \lambda h_2}{\theta_1}e^{h_2x}, 0 < x < x_*$$

$$f_2(x) = d_{11}h_1\frac{\theta_0}{\theta_2 - \mu_0h_1}e^{h_1x} + d_{12}h_2\frac{\theta_0}{\theta_2 - \mu_0h_2}e^{h_2x}, 0 < x < x_*$$

The expression of the average social benefit $B(x)$ per unit time can be obtained after manipulating. Finally, to solve the optimization problem of $\arg\max\{B(x_*)\}$, then the socially optimal threshold x_* per unit time can be obtained.

5. Optimal analysis

Based on the theoretical analysis results mentioned above, this section takes the fully observable case as an example to discuss the sensitivity of social benefits per unit time to changes in system parameters. Furthermore, the optimal social benefits are provided using the SOA algorithm.

When $\lambda > \mu_1$, the social benefits per unit time, $B\left(x_*(0),x_*(1),x_*(2)\right)$, are only related to $x_*(0)$ and $x_*(1)$, and can be simplified to $B\left(x_*(0),x_*(1)\right)$. Analyze the changes in the social benefits function with respect to the thresholds $x_*(0)$ and $x_*(1)$. According to Theorem 3, both λ_e and $E(x)$ increase with the increases in . $x_*(0)$. and $x_*(1)$. The expression for $B\left(x_*(0),x_*(1)\right)$ is relatively complex, making its monotonicity difficult to determine. Assuming that $\theta_0 = 2$, $\theta_1 = 1$, $\theta_2 = 1.5$, $\lambda = 2$, $\mu_0 = 1$, $\mu_1 = 1.5$, $\alpha = 0.2$, $R = 10$, $C = 1$, the three-dimensional variation of $B\left(x_*(0),x_*(1)\right)$ with $x_*(0)$ and $x_*(1)$ can be observed, as shown in Figure 1.From Figure 1, it can be seen that given $x_*(1)$, $B\left(x_*(0),x_*(1)\right)$ first decreases and then increases with $x_*(0)$. Conversely, given $x_*(0)$, increases slowly with the increase in $x_*(1)$.

When $\lambda = \mu_1$, the social benefits per unit time, $B\left(x_*(0),x_*(1),x_*(2)\right)$, are only related to $x_*(0)$ and can be simplified to $B\left(x_*(0)\right)$. Assuming that $\theta_0 = \theta_1 = \theta_2 = 2$, $\lambda = 2$, $R = 10$, $C = 3$, and the optimal social threshold is $x_*(0) = 5$, the curves of $B\left(x_*(0),x_*(1)\right)$ with changes in μ_0 and μ_1 can be observed, as shown in Figure 2.

From Figure 2, it can be seen that as the outflow rates μ_0 and μ_1 increase, the social benefits per unit time show an upward trend, and the changes are significant. This is because when the outflow rate of the buffer increases, the sojourn time of the fluid in the system is shortened, making the fluid more willing to enter the system. Moreover, increasing the outflow rate μ_0 during the incomplete fault state can accelerate the growth rate of social benefits per unit time. On the other hand, assuming $\theta_0 = \theta_1 = \theta_2 = 2$, $\mu_0 = 2$, $R = 10$, $C = 3$, the changes in social benefits with the balking threshold $x_*(0)$ are analyzed. As shown in Figure 3, when the inflow rate λ is fix $B\left(x_*(0),x_*(1)\right)$ ed, the social benefits increase first and then decrease with the increase of the threshold $x_*(0)$. For different inflow rates λ, at the same threshold $x_*(0)$, the social benefits per unit time increase with the increase in λ. When the inflow rates are $\lambda = 3$, $\lambda = 3.5$, and $\lambda = 4$, the maximum social benefits per unit time are obtained at $x_*(0) = 2.4$, $x_*(0) = 2.6$, and $x_*(0) = 2.7$, with maximum values of 14.03, 16.17, and 18.26, respectively. Furthermore, for $\lambda = 4$, using the SOA optimization algorithm to find the approximate global optimal balking threshold, the optimal social threshold $x_*(0) = 0.87$ is obtained, with an optimal social benefit per unit time of 14.93.

When $\lambda < \mu_1$, $B\left(x_*(0),x_*(1),x_*(2)\right)$ is only related to $x_*(0)$ and can be simplified to $B\left(x_*(0)\right)$. Assuming that $\theta_0 = \theta_1 = \theta_2 = 2$, $\mu_0 = 2$, $R = 10$, $C = 3$, as shown in Figure 4, when the inflow rate λ is fixed, the social benefits first increase and then decrease with the increase of the threshold $x_*(0)$. For different inflow rates λ, at the same threshold $x_*(0)$, the social benefits per unit time increase with the increase in λ. When the inflow rates are $\lambda = 3.4$, $\lambda = 3.6$, and $\lambda = 3.8$, the maximum social benefits per unit time are obtained at $x_*(0) = 3.6$, $x_*(0) = 3.4$, and $x_*(0) = 3.1$, with maximum values of 18.21, 18.77, and 19.26, respectively. Furthermore, for $\lambda = 3$ and $\mu_1 = 5$, using the SOA optimization algorithm to find the approximate global optimal balking threshold, the optimal social threshold $x_*(0) = 2.53$ is obtained, with an optimal social benefit per unit time of 13.79.

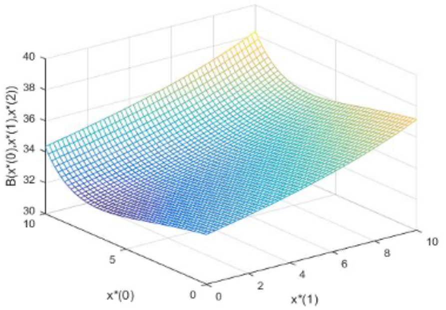

Figure 1 *The change of* $B(x_*(0), x_*(1))$ *with* $x_*(0)$ *and* $x_*(1)$ *given* $\lambda > \mu_1$.

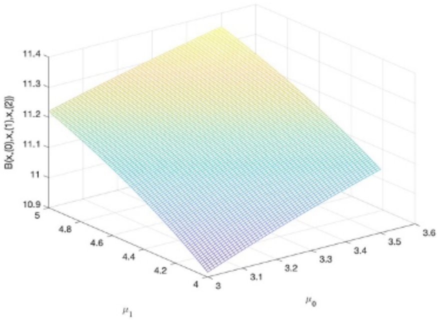

Figure 2 *The change of* $B(x_*(0), x_*(1))$ *with* μ_0 *and* μ_1 *given* $\lambda = \mu_1$.

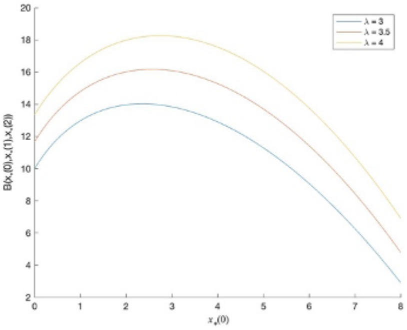

Figure 3 *The change of* $B(x_*(0), x_*(1))$ *with* $x_*(0)$ *given* $\lambda = \mu_1$.

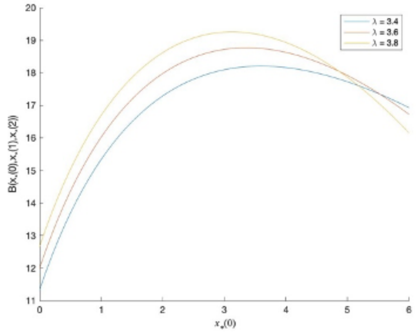

Figure 4 *The change of* $B(x_*(0), x_*(1))$ *with* $x_*(0)$ *given* $\lambda < \mu_1$.

6. Conclusion

This paper constructs a fluid queueing model with incomplete fault and repair delay. Under fully observable and almost observable cases, the conditional expectation theorem is used to derive the average sojourn time of fluid in the system. The steady-state distribution of the fluid level is calculated using the eigenvector method for systems of linear differential equations with constant coefficients. Individual and social utility models are constructed separately, and the individual equilibrium balking threshold and the socially optimal threshold of the fluid are discussed from an economic perspective. Finally, numerical examples are used to discuss the effects of thresholds and inflow rates on the average social benefits per unit time. There is still considerable scope for research on the equilibrium strategy analysis of fluid queueing models. Depending on the needs of practical applications, the utility function can be extended to more general cases. For example, when there is significant uncertainty in the sojourn time, a mean-variance utility function can be considered, which is widely used in finding the efficient frontier of investment portfolios.

Acknowledgments

The authors would like to thank the anonymous reviewers for their constructive comments and suggestions.This work was supported by National Natural Science Foundation [grant number 62171143] and Hebei Natural Science Foundation [grant number G2024203008].

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(Received July 2024; accepted January 2026)