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Optimal Ordering and Advertisement Policies for a Two-Warehouse Inventory System with Advance Sales and Product Return Guarantee

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Keywords

Inventory
Advance sales
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Return products

Abstract.

In the real market, competition is becoming more intense. Companies must create customer value. They must also clearly and persuasively communicate that value to target customers. In practice, advertisement plays a key role in stimulating demand. This research incorporates the frequency of advertisement into the demand function. In order to develop and maintain a competitive advantage, the retailer offers an appreciation period in which customers can ask to return products for any reason. In addition, by providing advance sales, retailers extend the sales period and increase market share. In this article, we provide optimal ordering and advertisement policies for a two-warehouse inventory system with advance sales and product return guarantee. We first establish a proper model and then provide an easy-to-use method to obtain optimal ordering and advertisement policies for the retailer to achieve maximum total profit. Finally, numerical examples are given to illustrate the solution procedure, followed by a sensitivity analysis that examines the effect on the optimal solution of changes in the values of certain parameters.

1. Introduction

Frequently, retailers offer consumers return guarantees to reduce the risk borne by consumers who cannot fully evaluate a product before purchase. Li et al. [11] studied direct distributors' pricing strategy, return policy, and quality policy under four scenarios. The scenarios included situations where consumer demand is sensitive to the price or return policy and scenarios when returns are also sensitive to return policy or quality. Kulkarni et al. [10] assumed that the retailer could order twice prior to the start of

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the selling season and that the manufacturer sets two alternative returns policies, and the retailer chooses any one of them. Taleizadeha et al. [26] developed two pricing and inventory models for two competing supply chains under composite coordinating strategies. Each chain contains one manufacturer and multiple non-competing retailers. The manufacturer receives raw materials from an outside supplier, transforms them into finished products, and then sells these to retailers to satisfy customers' demands. Ülkü et al. [28] introduced a variant of the classical single-period inventory (newsvendor) model with returns, in which heterogeneous consumers decide, based on their post-purchase valuation of the product, whether to return the product after using it. Sanni et al. [23] studied a reverse logistics EOQ model and discussed the effect of the reverse flow of products as an inventory problem. Liu and Chen [14] studied a retailer's Money-back guarantees (MBGs) policy with dynamic pricing of limited inventory. A key decision for the retailer is to decide whether to offer MBGs.

In practice, advertisement plays a key role in stimulating demand. Yadav et al. [30] investigated an economic order quantity model with backorder by taking imprecise demand rate with dependence upon the frequency of advertisement. Rabbani et al. [19] provided an integrated model for dynamic pricing and inventory control of non-instantaneous deteriorating items in which the demand rate is dependent on the frequency of advertisement in each replenishment cycle. Manna et al. [15] presented an economic production quantity model with an imperfect production system and advertisement-dependent demand. The advertisement rate was assumed to be a function of time which increased with respect to time at a decreasing rate. Rabbani et al. [20] provided an integrated model for dynamic pricing and inventory control of non-instantaneous deteriorating items. The demand rate is a function of advertisement and changes in price over time. Shaikh et al. [25] developed an inventory model for a deteriorating item with variable demand dependent on the selling price and frequency of advertisement of the item under the financial trade credit policy. San-José et al. [22] developed the optimal policy for an inventory system with demand dependent on price, time and frequency of advertisement.

In addition, advance sales policies are widely used by retailers (such as G-music.com.tw, Amazon.com, and Eslitebooks.com). Tsao [27] considered retailers' promotion and replenishment policies with an advance sales discount under the supplier's and retailer's trade credits and presented an algorithm to simultaneously determine the optimal promotion effort and replenishment cycle time. Chen and Cheng [2] established an inventory model for retailers who simultaneously receive a permissible delay in payments from suppliers while offering advanced sales to customers. Dye and Hsieh [7] established an advance sales system in which each sales cycle is divided into an advance sales period and a spot sales period for retailers' deteriorating items. During the advance sales period, customers were required to make reservations while customers with reservations made cancellations. Cheng and Ouyang [4] established an inventory model with price-dependent demand for a retailer who simultaneously receives trade credit from a supplier and offers advance sales and an appreciation period to customers. Youjun et al. [32] provided a non-multicycle inventory model for deteriorating items with advance sales over the finite time level and used a dynamic programming method to solve, in which the demand rate is time-varying and price-dependent, allowing cancellation but requiring payment of a cancellation fee. Seref et al. [24] studied a retailer's inventory and pricing decisions in an advance selling scenario that involves strategic consumers who consider the possible unavailability of inventory during spot sales. Cheng et al. [3] established an inventory model for deteriorating items with a return period and price-dependent demand for a retailer that offers two-phase advance sales to customers. Duary et al. [5] constructed advance and delay in payments with the price-discount inventory model for deteriorating items under capacity constraint and partially backlogged shortages.

In practice, due to the limited capacity of the owned warehouse, one additional warehouse is required while a large amount of stock is held. Yang [31] proposed an alternative model for determining the optimal replenishment cycle for a two-warehouse inventory problem under inflation, in which the inventory deteriorates at a constant rate over time and shortages are allowed. Dye et al. [6] developed a deterministic inventory model for deteriorating items with two warehouses. A rented warehouse is used when the ordering quantity exceeds the limited capacity of the owned warehouse, and it is assumed that deterioration rates of items in the two warehouses may be different. Hsieh et al. [8] developed a deterministic inventory model for deteriorating items with two warehouses. They allowed for shortages and complete backlogging and assumed that the inventory costs (including holding cost and deterioration cost) in RW are higher than that in OW. Sana [21] established an interesting multi-item EOQ (Economic Order Quantity) model in which time-varying demand was influenced by enterprises' initiatives such as advertising media and salesmen's effort. It was developed for deteriorating and ameliorating items with capacity constraints for a storage facility. Wang et al. [29] considered a single-manufacturer-single-buyer supply chain problem in which the manufacturer produced a single deteriorating product and delivered it to the buyer based on a consignment policy. An integrated inventory control model, jointly determining the manufacturer's production batch and the replenishment lot sizes, is proposed to minimize the manufacturer's total cost per unit time. Liao et al. [13] dealt with a deterministic order-level inventory model for deteriorating items with finite warehouse capacity and addressed the conditions of permissible delay in payments in the above model to respond to the prevailing economic conditions. Liao et al. [12] developed an inventory model by considering two levels of trade credit, limited storage capacity, and assumed that the retailer can delay incurring interest charges on the unpaid and overdue balance due to the difference between interest earned and interest charged. Ouyang et al. [17] proposed an integrated inventory model with capacity constraints and a permissible delay payment period that is order-size dependent. In addition, the unit production cost, which is a function of the production rate, was also considered. Panda1 et al. [18] investigated a two-warehouse inventory model for a deteriorating item using a credit policy approach. Demand was dependent on the frequency of advertisement, price, and stock. Chandra. [1] constructed the two warehouse inventory model for deteriorating items with stock dependent demand under permissible delay in payment. Meenua [16] studied the inventory policy for deteriorating items with two- warehouse and effect of carbon emission.

In this paper, we discuss inventory issues, including return products, advance sales, and advertisement. We establish a two-warehouse inventory system with advance sales and a product return guarantee and consider a scenario in which the demand is affected by the price and advertisement frequency. The paper is organized as follows. The related literature is reviewed in Section 1, notation and assumptions are introduced in Section 2. The model is developed in Section 3. The theoretical results and algorithm to obtain the optimal solutions are presented in Section 4. In Section 5, a numerical example is given, and a sensitivity analysis is undertaken to illustrate the proposed model. Finally, conclusions and directions for future research are given in Section 6

2. Notation and Assumptions

The mathematical model in this paper is developed on the following notation and assumptions.

Notation:

p unit selling price, a decision variable.

c unit purchase cost, where $c < p$.

s ordering cost per order.

c_a cost of each advertisement.

W the capacity of the owned warehouse (denoted by OW).

k cost of implementing advance sales.

h_1 unit holding cost per unit time in OW.

h_2 unit holding cost per unit time in a rented warehouse (denoted by RW), where $h_2 > h_1$.

h_3 unit holding cost per unit time of return product, where $h_3 \leq h_1$.

I_e retailer's interest earned per dollar per unit time.

N appreciation period.

δ advance sales discount rate (i.e., all products are $\delta \times 100\%$ off during the advance sales period), where $0 \leq \delta \leq 1 - c/p$.

θ the pre-order cancellation rate, where $0 \leq \theta < 1$.

ρ return rate during appreciation period, where $0 \leq \rho < 1$.

ω the rate for non-defective return goods, where $0 \leq \omega < 1$.

γ sales discount rate for non-defective return products, where $0 \leq \gamma < 1$.

t_1 advance selling period in which order cancellation is permitted.

t_2 advance selling period ($t_2 - t_1$ is advance selling period in which order cancellation is non-permitted).

t_3 the time at which the inventory level reaches zero in RW.

A the frequency of advertisement, positive integer and is a decision variable.

T sales period ($T - t_2$ is the spot selling period), a decision variable.

D	demand rate.
Q	order quantity, a decision variable.
p^*	the optimal selling price.
A^*	the optimal frequency of advertisement.
T^*	optimal sales period.
Q^*	optimal order quantity.
$Z(p, A, T)$	total profit.
Z^*	maximum total profit, i.e., $Z^* = Z(p^*, A^*, T^*)$.

Assumptions:

1. The inventory system functions for a single item in a single season.
2. Replenishment occurs instantaneously at an infinite rate.
3. Shortages are not permitted.
4. The owned warehouse (OW) has a fixed capacity of W units; the rented warehouse (RW) has unlimited capacity.
5. The goods of OW are consumed only after consuming the goods kept in RW.
6. All products are $\delta \times 100\%$ off during the advance sales period $[0, t_2[$.

   Customers who accept advance sales offer must pay for the pre-committed orders. In this situation, the retailer can use the early cash income to generate interest earned.
7. The advance sales period $[0, t_2]$ can be divided into two parts: $[0, t_1]$ and $[t_1, t_2]$. In period $[0, t_1]$, customers with reservations might cancel their reservations. The cancellation rate is θ. However, in the other period $[t_1, t_2]$, customers with reservations cannot cancel their orders.
8. The demand rate is a function of the selling price p and frequency of advertisement A and given as (see, for example, Kocabıyıkoğlu and Popescu [9], Rabbani et al. [20]) $D \equiv D(p, A) = (a - bp)(1 + A)^{\lambda}$, where λ is the shape parameter of the advertisement $0 < \lambda < 1$, $a > 0$, $b > 0$ and $0 < p < a/b$.
9. To guarantee the optimal solution exists, we assume that the maximum return quantity for items in OW, ρW, is less than the demand rate $(a - bp)(1 + A)^{\lambda}$; that is, $\rho w < (a - bp)(1 + A)^{\lambda}$.
10. Retailers offer consumers return guarantees during the appreciation period N. Customers can make a request to return products for any kind of reason during the appreciation period N.
11. If customers cancel their orders or return products, the retailer will refund the money.
12. All return products are divided into non-defective products and defective products. At time $T + N$, the non-defective return products are sold in a single batch at a discounted price $(1 - \gamma)\rho$, where γ is the discount rate and the defective return products are discarded.

3. Mathematical Formulation

This article establishes a two-warehouse inventory system, where the replenishment cycle is divided into two sales periods: one is the advance sales period $[0, t_2]$ and the other is the spot sales period $[t_2, T]$. In the advance sales period $[0, t_1]$, customers with reservations can cancel their reservations. The cancellation rate is θ. However, in the other advance sales period $[t_1, t_2]$, customers with reservations cannot cancel the orders. On the other hand, when the customers receive an order, they may make a request to return products for any kind of reason during the appreciation period. The return rate is ρ. In the advance sales period $[0, t_2)$, the retailer offers $\delta \times 100\%$ discount rate to customers. Hence, the unit selling price is $p_1 = (1-\delta)p$, and the market demand rate is $D_1(p_1, A) = D(p_1, A) = (a - bp_1)(1 + A)^{\lambda}$. In the spot sales period $[t_2, T]$, the unit selling price is $p_2 = p$, and the demand rate is $D_2(p_2, A) = D(p_2, A) = (a - bp_2)(1 + A)^{\lambda}$. For convenience, D_i (p_i, A) and D_i will be used interchangeably, for $i = 1, 2$. It can be shown that the total accumulative advance sales quantity in the advance sales period is $(1-\theta)D_1t_1 + D_1(t_2 - t_1) = D_1(t_2 - \theta t_1)$. The maximum inventory level during the spot sales period is $D_2(T - t_2)$. Therefore, the ordering quantity in the replenishment cycle is given by

$$Q = D_1(t_2 - \theta t_1) + D_2(T - t_2). \tag{3.1}$$

Figure 1 displays the behavior of inventory level.

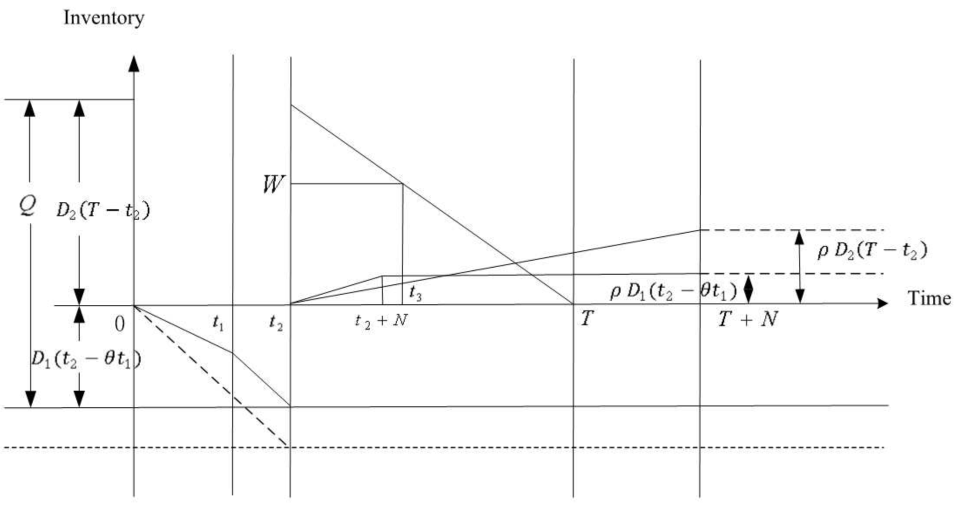

Figure 1: The retailer's inventory level.

The objective is to maximize the retailer's total profit. Total profit is total sales revenue plus the interest earned and minus the total relevant cost (which includes the cost of implementing advance sales, advertisement cost, ordering cost, purchasing, and holding costs). These components are calculated as follows:

(a) Sales revenue SR (which consists of sales revenue in advance sales period, sales revenue in spot sales period, sales revenue of non-defective return goods). These components are calculated as follows:

1. Sales revenue in advance sales period $[0, t_2]$ is $(1-\rho)p_1D_1(t_2-\theta t_1)$.
2. Sales revenue in spot sales period $[t_2, T]$ is $(1-\rho)p_2D_2(T-t_2)$.
3. Sales revenue of non-defective return goods is $(1-r)p\omega\rho Q = (1-r)p\omega\rho[D_1(t_2-\theta t_1)+D_2(T-t_2)]$. Thus,

$$SR = (1-\rho)p_1D_1(t_2-\theta t_1) + (1-\rho)p_2D_2(T-t_2) + (1-r)p\omega\rho[D_1(t_2-\theta t_1)+D_2(T-t_2)].$$

(b) Interest earned: in the advance sales period $[0, t_2]$, the unit selling price is p_1 and the customers who accept advance sales offer must pay for the pre-committed orders, with an interest at a rate of I_e. The retailer's advantage gain from early payment is

$$IE = \frac{I_e p_1 D_1}{2}[(1-\theta)t_1^2 + (t_2-t_1)^2].$$

(c) Cost of implementing advance sales $IC = k$,

(d) Advertisement cost $AC = c_aA$.

(e) Cost of placing an order $OC = s$.

(f) Cost of purchasing $PC = cQ = c[D_1(t_2-\theta t_1)+D_2(T-t_2)]$.

(g) Cost of carrying inventory HC (which consists of the cost of carrying inventory in the owned warehouse, cost of carrying inventory in the rented warehouse, cost of carrying inventory of return goods). These components are calculated as follows:

1. Cost of carrying inventory in the owned warehouse is $h_1W[\frac{1}{2}(T+t_3)-t_2]$,
2. Cost of carrying inventory in the rented warehouse is $\frac{h_2}{2}[D_2(t_3-t_2)^2]$.
3. Cost of carrying inventory of return goods is $h_3\rho[D_1(t_2-\theta t_1)(\frac{N}{2}+T-t_2)+\frac{1}{2}D_2(T-t_2)(T+N-t_2)]$. Thus,

$$HC = h_1W\Big[\frac{1}{2}(T+t_3)-t_2\Big] + \frac{h_2}{2}D_2(t_3-t_2)^2 + h_3\rho\Big[D_1(t_2-\theta t_1)\Big(\frac{N}{2}+T-t_2\Big) + \frac{1}{2}D_2(T-t_2)(T+N-t_2)\Big].$$

Therefore, the retailer's total profit Z is given by:

$$\begin{aligned} Z(p,A,T) &= SR + IE - IC - AC - OC - PC - HC \\ &= (1-\rho)p_1D_1(t_2-\theta t_1)+(1-\rho)p_2D_2(T-t_2)+(1-r)p\omega\rho[D_1(t_2-\theta t_1)+D_2(T-t_2)] \\ &\quad + \frac{I_ep_1D_1}{2}[(1-\theta)t_1^2+(t_2-t_1)^2] - k - c_aA - s - c[D_1(t_2-\theta t_1)+D_2(T-t_2)] \\ &\quad - h_1W\Big[\frac{1}{2}(T+t_3)-t_2\Big] - \frac{h_2}{2}[D_2(T-t_2)-W](t_3-t_2) \\ &\quad - h_3\rho\Big[D_1(t_2-\theta t_1)\Big(\frac{N}{2}+T-t_2\Big)+\frac{1}{2}D_2(T-t_2)(T+N-t_2)\Big] \end{aligned}$$

$$
\begin{aligned}
&= D_1(t_2-\theta t_1)\Big[(1-\rho)p_1+(1-\gamma)p\omega\rho-c-h_3\rho\Big(\frac{N}{2}+T-t_2\Big)\Big]\\
&\quad + D_2(T-t_2)\Big[(1-\rho)p_2+(1-\gamma)p\omega\rho-c-\frac{h_2}{2}(t_3-t_2-h_3)\frac{\rho}{2}(T+\mu-t_2)\Big]\\
&\quad + \frac{I_e p_1 D_1}{2}[(1-\theta)t_1^2+(t_2-t_1)^2]-h_1W\Big[\frac{1}{2}(T+t_3)-t_2\Big]\\
&\quad + \frac{h_2}{2}W(t_3-t_2)-k-c_aA-s
\end{aligned}
\tag{3.2}
$$

where $p_1=(1-\delta)p$, $p_2=p$, $D_i=(a-bp_i)(1+A)^{\lambda}$ for $i=1,2$, and $t_3=T-\frac{W}{D_2}$.

4. Theoretical Results

In this section, we present the solution procedure and find the optimal solution to the aforementioned case. First, for any given positive p and T, we temporarily relax the integer requirement on A and take the second-order derivative of $Z(A|p,T)$ with respect to A, which gives:

$$
\begin{aligned}
\frac{d^2Z(A|p,T)}{dA^2} &= (t_2-\theta t_1)\Big[(1-\rho)p_1+(1-r)p\omega\rho-c-h_3\rho\Big(\frac{N}{2}+T-t_2\Big)\Big]\frac{d^2D_1}{dA^2}\\
&\quad + (T-t_2)\Big[(1-\rho)p_2+(1-\gamma)p\omega\rho-\frac{h_2}{2}(t_3-t_2)-\frac{h_2\rho}{2}(T+N-t_2)\Big]\frac{d^2D_2}{dA^2}\\
&\quad - h_2(T-t_2)\frac{dD_2}{dA}\frac{dt_3}{dA}+\frac{I_ep}{2}[(1-\theta)t_1^2+(t_2-t_1)^2]\frac{d^2D_1}{dA^2}-\frac{h_2}{2}(T-t_2)\frac{d^2t_2}{dA^2}\\
&\quad + \frac{W}{2}(h_2-h_1)\frac{d^2t_3}{dA^2}.
\end{aligned}
\tag{3.3}
$$

We have

$$
\begin{aligned}
\frac{d^2D_1}{dA^2} &= (a-bp_1)\lambda(\lambda-1)(1+A)^{\lambda-2},\\
\frac{d^2D_2}{dA^2} &= (a-bp_2)\lambda(\lambda-1)(1+A)^{\lambda-2},\\
\frac{dD_2}{dA}\frac{dt_3}{dA} &= W\lambda^2(1+A)^{-2}, \quad \text{and} \quad \frac{d^2t_3}{dA^2} = -W\lambda(\lambda+1)(a-bp_2)^{-1}(1+A)^{-\lambda-2}.
\end{aligned}
$$

Hence, Equation (3.3) can be rewritten as

$$
\begin{aligned}
&\frac{d^2Z(A|p,T)}{dA^2}\\
&= (t_2-\theta t_1)(a-bp_1)\lambda(\lambda-1)(1+A)^{\lambda-2}\Big[(1-\rho)p_1+(1-r)p\omega\rho-c-h_3\rho\Big(\frac{N}{2}+T-t_2\Big)\Big]\\
&\quad +(T-t_2)(a-bp_2)\lambda(\lambda-1)(1+A)^{\lambda-2}\Big[(1-\rho)p_2+(1-r)p\omega\rho-c-\frac{h_2}{2}(t_3-t_2)\\
&\quad -\frac{1}{2}h_3\rho(T+N-t_2)\Big]+\frac{1}{2}I_ep(a-bp_1)\lambda(\lambda-1)(1+A)^{\lambda-2}[(1-\theta)t_1^2+(t_2-t_1)^2]
\end{aligned}
$$

$$-\frac{1}{2}W^2\lambda(\lambda+1)(a-bp_2)^{-1}(1+A)^{-\lambda-2}(h_2-h_1)+\frac{h_2}{2}(T-t_2)W\lambda(\lambda+1)(1+A)^{-2}. \quad (3.4)$$

In Equation (3.4), the terms:

$(1-\rho)p_1+(1-r)p\omega\rho-c-h_3\rho\Big(\frac{N}{2}+T-t_2\Big)$

=unit sale revenue in advance sales period

+ unit sale revenue of non-defective return goods

− unit purchase cost − unit holding cost of return product for advance sale > 0,

$(1-\rho)p_2+(1-r)p\omega\rho-c-\frac{h_2}{2}(t_3-t_2)-\frac{1}{2}h_3\rho(T+N-t_2)$

=unit sale revenue in spot sales period+unit sale revenue of non-defective return goods

− unit purchase cost-unit holding cost in a rented warehouse

− unit holding cost of return product for advance sale > 0, $0<\lambda<1$, and $h_2>h_1$.

Furthermore, the only positive term in Equation (3.4) is the inventory related cost in the rented warehouse, which in general is less than the other related cost (e.g., advertisement cost, purchasing cost). Hence, we assume without loss of generality that $\frac{d^2Z(A|p,T)}{dA^2}<0$. Therefore, $Z(A|p,T)$ is a concave function of A, which implies searching for the optimal solution of A may be reduced to finding a local, optimal solution. Next, for a fixed A, we discuss how to find the optimal solution of (p,T). The optimal solution of (p,T) must satisfy the following equations:

$$\begin{aligned}
&\frac{\partial Z(A|p,T)}{\partial p}\\
&=(1-\rho)[(1-\delta)D_1-(1-\delta)^2pb(1+A)^{\lambda}](t_2-\theta t_1)+(1-\rho)[D_2-bp(1+A)^{\lambda}](T-t_2)\\
&\quad+(1-r)\omega\rho[D_1(t_2-\theta t_1)+D_2(T-t_2)]+(1-r)p\omega\rho[-b(1-\delta)(1+A)^{\lambda}(t_2-\theta t_1)\\
&\quad-b(1+A)^{\lambda}(T-t_2)]+\frac{I_e}{2}[(1-\delta)D_1-(1-\delta)^2pb(1+A)^{\lambda}][(1-\theta)t_1^2+(t_2-t_1)^2]\\
&\quad-c[-b(1-\delta)(1+A)^{\lambda}(t_2-\theta t_1)-b(1+A)^{\lambda}(T-t_2)]+\frac{h_1W^2}{2}b(a-bp)^{-2}(1+A)^{-\lambda}\\
&\quad+\frac{h_2}{2}b(1+A)^{\lambda}(T-t_2)(t_3-t_2)+\frac{h_2}{2}[D_2(T-t_2)-W]bW(a-bp)^{-2}(1+A)^{\lambda}\\
&\quad-h_3\rho[-b(1-\delta)(1+A)^{-\lambda}(t_2-\theta t_1)\Big(\frac{N}{2}+T-t_2\Big)\frac{-1}{2}b(1+A)^{\lambda}(T-t_2)(T+N-t_2)]=0,
\end{aligned} \quad (3.5)$$

where $t_3=T-\frac{W}{D_2}$
and

$$\begin{aligned}
&\frac{\partial Z(p,T|A)}{\partial p}\\
&=(1-\rho)p_2D_2+(1-\gamma)p\omega\rho D_2-cD_2-h_1W(1+1)-\frac{h_2}{2}[D_2(t_3-t_2)+D_2(T-t_2)-W]
\end{aligned}$$

$$-h_3\rho\Big[D_1(t_2-\theta t_1)+\frac{1}{2}D_2(T+N-t_2)+\frac{1}{2}D_2(T-t_2)\Big]=0, \tag{3.6}$$

where $t_3 = T - \frac{W}{D_2}$
simultaneously. In addition,

$$\frac{\partial^2 Z(p,T|A)}{\partial p^2} = -2(1-\rho)b(1-\delta)^2(1+A)^\lambda(t_2-\theta t_1)-2(1-\rho)b(1+A)^\lambda(T-t_2)$$
$$-2(1-r)W\rho b(1+A)^\lambda[(1-\delta)(t_2-\theta t_1)+T-t_2]-I_e b(1-\delta)^2(1+A)^\lambda[(1-\theta)t_1^2$$
$$+(t_2-t_1)^2]-W^2b^2(a-bp)^{-3}(1+A)^\lambda(h_2-h_1)+\frac{h_2}{2}b^2W(T-t_2)(a-bp)^{-2}, \tag{3.7}$$

$$\frac{\partial^2 Z(p,T|A)}{\partial p^2} = -hD_2 - h_3\rho D_2$$
$$= -D_2(h_2+h_3\rho)<0, \tag{3.8}$$

and

$$\frac{\partial^2 Z(p,T|A)}{\partial p\partial T} = (1-\rho)[D_2-pb(1+A)^\lambda]+(1-\gamma)\omega\rho D_2-(1-\gamma)p\omega\rho b(1+A)^\lambda$$
$$+bc(1+A)^\lambda+\frac{h_2}{2}b(1+A)^\lambda(t_3-t_2+T-t_2)+\frac{h_2}{2}D_2bw(a-bp)^{-2}(1+A)^{-\lambda}$$
$$+h_3\rho b(1+A)^\lambda\Big[(1-\delta)(t_2-\theta t_1+\frac{1}{2}(T+N-t_2)+\frac{1}{2}(T-t_2)\Big] \tag{3.9}$$

where $t_3 = T - \frac{W}{D_2}$.

Due to the complexity of the problem, we cannot easily identify whether the determinate of the Hessian matrix $H = \begin{bmatrix} \frac{\partial^2 Z}{\partial p^2} & \frac{\partial^2 Z}{\partial p\partial T} \\ \frac{\partial^2 Z}{\partial T\partial p} & \frac{\partial^2 Z}{\partial T^2} \end{bmatrix}$ is a positive value. However, we can check this condition by using software Mathematica 13.0 in numerical examples.

From the above results, we can develop the following algorithm to obtain the optimal ordering policy.

Algorithm

Step 1. Set $A = 1$.

Step 2. Determine p_A and T_A by solving Equations (3.5) and (3.6). Substituting p_A and T_A into Equation (3.2) to obtain $Z(p_A, A, T_A)$.

Step 3. Set $A = A + 1$. Repeat Step 2 and obtain $Z(p_A, A, T_A)$.

Step 4. If $Z(p_{A+1}, A+1, T_{A+1}) \geq Z(p_A, A, T_A)$ then go to Step 3, otherwise go to step 5.

Step 5. Set $Z(p^*, A^*, T^*) = Z(p_{A-1}, A_{A-1}, T_{A-1})$, then (p^*, A^*, T^*) is the optimal solution.

Step 6. Stop.

Once (p^*, A^*, T^*) is obtain, the optimal ordering quantity $Q^* = [a - b(1-\delta)p^*](1 + A^*)^\lambda(t_2 - \theta t_1) + (a - bp^*)(1 + A^*)^\lambda(T^* - t_2)$ follows.

5. Numerical examples

In this section, we give numerical examples to illustrate the above solution procedure.

Example 1. We consider a two-warehouse inventory system with the following data: $h_1 = 0.5$/unit/day, $h_2 = 0.6$/unit/day, $h_3 = 0.6$/unit/day, $c = \$15$/unit, $N = 7$, $w = 50$, $s = 20$, $k = 50$, $c_a = 50$, $\theta = 0.01$, $\delta = 0.15$, $\rho = 0.06$, $\omega = 0.4$, $\gamma = 0.2$, $t_1 = 20$, $t_2 = 30$, $a = 14$, $b = 0.2$, $\lambda = 1$ in appropriate units. Using the aforementioned algorithm, the determinate of the Hessian matrix $H = \begin{bmatrix} \frac{\partial^2 Z}{\partial p^2} & \frac{\partial^2 Z}{\partial p \partial T} \\ \frac{\partial^2 Z}{\partial T \partial p} & \frac{\partial^2 Z}{\partial T^2} \end{bmatrix} = 41.34$, is a positive value, furthermore, we obtain the optimal solution is $(p^*, T^*, A^*) = (48.5233, 215.435, 18)$ and $Z(p^*, T^*, A^*) = \$13174.3$. The optimal time at which the inventory level reaches zero in the rented warehouse $t_3^* = T^* - \frac{W}{(a-bp^*)(1+A^*)^\lambda} = 193.962$ and the optimal order quantity $Q^* = [a - b(1-\delta)p^*](1+A^*)^\lambda(t_2 - \theta t_1) + (a - bp^*)(1+A^*)^\lambda(T^* - t_2) = 1088.87$ units.

Example 2. Using the same data as in Example 1, we study the influence of a single parameter change to the optimal solution. The numerical results are presented in Table 1.

Table 1: Effects of parameters on the optimal solution.

	parameters	p^*	T^*	A^*	Q^*	$Z(p^*, A^*, T^*)$
h_1	0.3	49.0009	236.290	18	997.20	15021.1
	0.5	48.5233	215.435	18	1088.87	13174.3
	0.7	48.0836	195.177	17	1007.64	11526.9
	0.9	47.6836	175.253	15	920.58	10072.3
h_2	0.4	48.9088	294.928	23	1461.42	18021.7
	0.6	48.5233	215.435	18	1088.87	13174.3
	0.8	48.2369	165.321	14	848.99	10315.1
	1	48.0686	130.955	11	680.93	8512.2
h_3	0.4	48.4272	244.049	20	1237.26	14841.9
	0.6	48.5233	215.435	18	1088.87	13174.3
	0.8	48.6151	192.246	15	966.27	11863.3
	1	48.7171	173.640	14	871.26	10809.3
δ	0.15	48.5233	215.435	18	1088.87	13174.3
	0.20	48.7704	215.042	17	1089.26	13006.5
	0.25	48.9886	214.696	17	1093.79	12769.0
	0.30	49.1724	214.174	17	1099.25	12459.8
θ	0.01	48.5233	215.435	18	1088.87	13174.3
	0.03	48.5148	215.362	17	1083.98	13125.7
	0.05	48.5109	215.525	17	1082.37	13077.1
	0.07	48.5070	215.688	17	1080.76	13028.6
ρ	0.03	48.3235	424.344	37	2176.76	24500.5
	0.06	48.5233	215.435	18	1088.87	13174.3
	0.09	48.7582	146.456	11	734.43	9559.5
	0.12	42.9253	99.4355	12	637.94	7122.18

Table 1 reveals that a higher value of holding cost in the owned warehouse h_1 results in a lower optimal selling price p^*, the optimal replenishment cycle time T^*, the optimal frequency of advertisement A^* and the optimal total profit, $Z(p^*, A^*, T^*)$. A higher value of holding cost in the owned warehouse h_1 results in a higher optimal order quantity Q^* when h_1 is less than or equal to 0.5 and a lower optimal order quantity Q^* when h_1 is greater than or equal to 0.7. This implies that the retailers must reduce the order quantity to avoid a higher holding cost in the owned warehouse. A higher value of holding cost in a rented warehouse h_2 results in a lower optimal selling price p^*, the optimal replenishment cycle time T^*, the optimal frequency of advertisement A^*, the optimal order quantity Q^* and the optimal total profit, $Z(p^*, A^*, T^*)$. This implies that retailers must reduce their replenishment cycle time and order quantity to avoid a higher holding cost in a rented warehouse. A higher holding cost value of return product h_3 results in a higher optimal selling price p^*, but lower values for optimal replenishment cycle time T^*, the optimal frequency of advertisement A^*, the optimal order quantity Q^*, the optimal total profit, $Z(p^*, A^*, T^*)$. This implies that retailers must reduce their replenishment cycle time and their order quantity to avoid a higher holding cost of return products. A higher value of advance sales discount rate δ results in a higher optimal selling price p^* and the optimal order quantity Q^*, but lower values for the optimal replenishment cycle time T^* and the optimal total profit, $Z(p^*, A^*, T^*)$. This implies that the retailers will increase their order quantity to meet the increased demand due to a higher value of advance sales discount rate. A higher value of pre-order cancellation rate θ results in lower values for optimal selling price p^*, the optimal order quantity Q^* and the optimal total profit, $Z(p^*, A^*, T^*)$. A higher value of return rate ρ results in lower values for optimal replenishment cycle time T^*, the optimal order quantity Q^* and the optimal total profit, $Z(p^*, A^*, T^*)$.

6. Conclusions

In this article, we incorporate some real phenomena in the real market for instance advance sales policies which are widely used by retailers such as G-music.com.tw, Amazon.com, and Eslitebooks.com and incorporate the frequency of advertisement into the demand function and discuss inventory issues, including return products. In order to keep and sustain a competitive advantage, the retailer offers an appreciation period in which customers can make a request to return products for any kind of reason. We establish a two-warehouse system with advance sales and product return guarantee and consider a scenario in which the demand is affected by the price and advertisement frequency. We provide an easy and useful algorithm to find the optimal ordering and advertisement policies. Finally, numerical examples are given to illustrate the solution procedure. The results of the sensitivity analysis show that the retailers must reduce the order quantity to avoid higher holding costs in the owned warehouse. The retailers must reduce their replenishment cycle time and their order quantity to avoid a higher holding cost in a rented warehouse. The retailers need to reduce their replenishment cycle time and their order quantity to avoid a higher holding cost for returned products. However, the retailers should increase their order quantity to meet the increased demand due to a higher value of advance sales discount rate. In future research, our model can be extended in several ways. A future avenue of investigation would be to consider a scenario in which retailers incorporate some hidden inventory costs, such as transportation costs.

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