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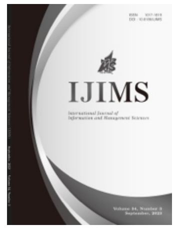

International Journal of
Information and Management Sciences

35 (2024), 63-78. DOI:10.6186/IJIMS.202403_35(1).0004

Revisiting Inventory Models: A Comparative Analysis of Two Trade Credit Policies

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Keywords

Inventory;
Time-varying demand;
Deterioratingitems; Trade credit.

Abstract

In this paper, we revisit the inventory models featuring two trade credit policies as originally proposed by Hsieh, Chang, Dye, and Weng in their work published in the International Journal of Information and Management Sciences (2009, Vol. 20, pp. 191-204). Our analysis focuses on the computation of interest earned within these models, revealing certain questionable outcomes. Subsequently, we offer a revised perspective aimed at addressing the identified issues, thereby contributing to the enhancement of their significant contributions in the field.

1. Introduction

The study conducted by Hsieh et al. (2009) introduced a novel inventory model featuring two trade credit policies, extending previous models to encompass time-varying demand and deterioration. However, a closer examination of their derived interest earned for accumulated sold items prior to the customer's trade credit reveals certain outcomes that warrant scrutiny. Notably, eight papers have cited the work of Hsieh et al.(2009) in their references, namely: Bakker et al. (2012), Kumar and Aggarwal (2012), Teng et al. (2012), Mahata and Mahata(2014), Chaudhary et al. (2018), SundaraRajan and Uthayakumar (2017), Tripathi(2019), and Gupta et al. (2020).

Among these, Bakker et al. (2012) undertook a comprehensive review of 241 inventory model papers concerning deteriorated items from 2001 onwards. Their evaluation of Hsieh et al. (2009) acknowledged the adoption of a trade credit policy involving supplier-retailer permissible delay (M) and retailer-customer trade credit period (N), ultimately concluding that Hsieh et al. (2009) utilized an extensive mathematical analysis to establish the existence of a unique, globally optimal minimal cost solution. However, our scrutiny disputes the assertion made by Bakker et al. (2012). Our study identifies a critical oversight in Hsieh et al.'s (2009) inventory models,

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which results in the neglect of certain items sold within the [0, N] timeframe, consequently questioning the validity of their models. As part of our contribution, we offer enhancements to rectify these discrepancies in their mathematical analysis.

Kumar and Aggarwal (2012) introduced a distinct inventory model centered on demand following an innovation diffusion process within a permissible payment delay context. While they acknowledged Hsieh et al.'s (2009) consideration of optimal lot sizes under trade credit financing amidst fluctuating demand and deterioration, their evaluation remained cursory. Teng et al. (2012) investigated an inventory model under non-decreasing demand, omitting consideration of retailer-customer trade credit, thereby distinguishing their model from Hsieh et al.'s (2009). Similarly, Mahata and Mahata (2014) explored inventory systems with quadratic demand devoid of deterioration and retailer-customer trade credit, further diverging from Hsieh et al.'s (2009) framework.

Chaudhary et al. (2018) conducted an extensive review of 418 inventory-related papers published between 1990 and 2016. In their classification, Hsieh et al. (2009) were grouped under "Time or inventory-dependent deterioration rate." However, their review offered no further insights into Hsieh et al.'s (2009) work. SundaraRajan and Uthayakumar (2017) proposed an inventory model incorporating permissible payment delay and exponentially escalating holding costs. However, they refrained from delving into the solution methodology adopted by Hsieh et al. (2009). Tripathi (2019) presented a deterministic Economic Order Quantity (EOQ) model with quadratic demand and completely backlogged shortages. Despite mentioning Hsieh et al. (2009) in their introduction, they did not pursue a comprehensive examination. Gupta et al. (2020) devised an inventory system encompassing time-varying deterioration, partially backlogged shortages, and permissible payment delay within a dual-warehouse framework. Yet, their discussion of Hsieh et al. (2009) remained introductory in nature, lacking detailed analysis.

Drawing from our analysis, none of the cited works addressed the identified questionable results that form the crux of our study. This paper is organized as follows: Section 2 reviews notation and assumptions, Section 3 recaps the average total cost across three distinct domains, Section 4 provides an exhaustive discussion on interest earned, and Section 5 addresses typographical errors in the first derivatives of the objective functions. In Section 6, we demonstrate the smooth connectivity of the objective function across adjacent domains. Section 7 contends that a comprehensive discussion of the second derivatives of the three sub-objective functions, as conducted in Hsieh et al. (2009), is superfluous. Furthermore, we streamline Hsieh et al.'s (2009) development by presenting three Theorems instead of their elaborate propositions. In Section 8, we assess two numerical examples proposed by Hsieh et al. (2009), and finally, we conclude our study in Section 9.

2. Notation and Assumptions

To be compatible with Hsieh et al. (2009), we adopt the same notation and assumptions as them.

Notation:

$f(t)$ =the demand rate is a continuous function of time t and increases at an increasing rate, i.e. satisfies$f'(t) > 0$ and $f''(t) > 0$.

A =the setup cost per order.

c =theunit purchasing cost.

p =theunit selling price, with$p > c$.

h =the holding cost excluding interest charge, \$/ per unit/year.

I_e =theinterest earned per \$ per year.

I_k =theinterest charged per \$ in stocks per year the supplier.

M =the retailer's trade credit period offered by the supplier in years.

N =the customer's trade credit period offered by the retailer in years, with $N \leq M$.

T =the replenishment time interval, with $T > 0$.

$AC(T)$ =the average total inventory cost per unit time that will be piece-wisely defined for three sub-domains: $M \leq T$, $N \leq T \leq M$, and $T \leq N$.

$AC_1(T)$ =the average total inventory cost per unit time, with $M \leq T$.

T_1^* =the local minimum for $AC_1(T)$, with $M \leq T$.

$AC_1(T_1^*)$ =the minimum value for $AC_1(T)$, with $M \leq T$.

$AC_2(T)$ =the average total inventory cost per unit time, with $N \leq T \leq M$.

T_2^* =the local minimum for $AC_2(T)$, with $N \leq T \leq M$.

$AC_2(T_2^*)$ =the minimum value for $AC_2(T)$, with $N \leq T \leq M$.

$AC_3(T)$ =the average total inventory cost per unit time, with $T \leq N$.

T_3^* =the local minimum for $AC_3(T)$, with $T \leq N$.

$AC_3(T_3^*)$ =the minimum value for $AC_3(T)$, with $T \leq N$.

Assumptions:

1. The inventory system involves only one item.

2. The replenishment occurs instantaneously at an infinite rate.

3. The items deteriorate at a varying rate of deterioration $\theta(t)$, with $\theta'(t) \geq 0$ and $0 < \theta(t) \leq 1$. Here, $\theta'(t)$ denotes the first derivative of $\theta(t)$ with respect to t. Note that $\theta'(t) \geq 0$ means that the deterioration rate is non-decreasing over time.

$g(t)$is an auxiliary function, with $g(t) = \int_0^t \theta(u)\, du$

4. When $M \leq T$, the account is settled at $T = M$ and the retailer starts paying for the interest charges on the items in stock with rate I_k. When $T < M$, the account is settled at $T = M$ and the retailer does not need to pay any interest charge.

5. The retailer can accumulate revenue and earn interest after his/her customer pays for the amount of purchasing cost to the retailer until the end of the trade credit period offered by the supplier. That is, the retailer can accumulate revenue and earn interest during the period $[N, M]$ with a rate I_e under the condition of trade credit.

3. Review of Hsieh et al. (2009)

We have extracted their outcomes concerning the average total cost per unit time within three distinct domains: Case 1 (Eq. 3-1), Case 2 (Eq. 3-2), and Case 3 (Eq. 3-3). For a comprehensive understanding of the derivations underlying their inventory models, we direct readers to the original work by Hsieh et al. (2009). The ensuing section (Section 4) delves into an elucidation of the debatable findings.

In the context of Case 1, wherein , Hsieh et al. (2009) deduced that

$$AC_1(T)=\frac{1}{T}\left\{A+c\int_0^T\left[e^{g(t)}-1\right]f(t)dt+h\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)dudt\right.$$
$$\left.+cI_k\int_M^T e^{-g(t)}\int_t^T e^{g(u)}f(u)dudt-pI_e\int_N^M(M-t)f(t)dt\right\} \tag{3-1}$$

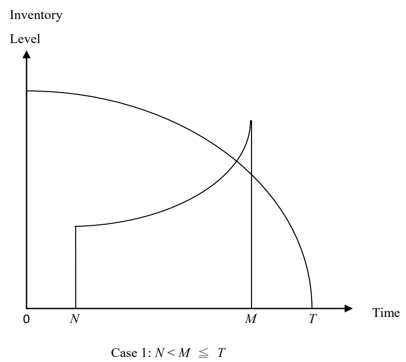

Figure 1 *Reproduction of Figure 1, Case 1, of Hsieh et al. (2009)*

For Case 2 with $N \le T \le M$, Hsieh et al. (2009) derived that

$$AC_2(T)=\frac{1}{T}\left\{A+c\int_0^T\left[e^{g(t)}-1\right]f(t)dt+h\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)dudt\right.$$
$$\left.-pI_e\left[\int_N^T(T-t)f(t)dt+(M-T)\int_0^T f(t)dt\right]\right\} \tag{3-2}$$

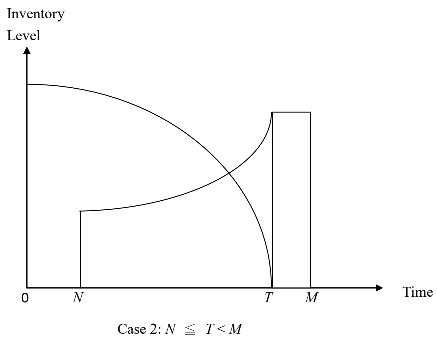

Figure 2 *Reproduction of Figure 1, Case 2, of Hsieh et al.(2009)*

For Case 3 with $0 < T \le N$, Hsieh et al. (2009) showed that

$$AC_3(T) = \frac{1}{T}\left\{A + c\int_0^T \left[e^{g(t)} - 1\right] f(t)dt + h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)dudt \right.$$
$$\left. -pI_e(M-T)\int_0^T f(t)dt\right\} \quad (3\text{-}3)$$

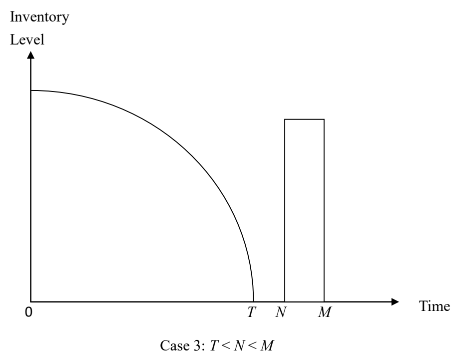

Figure 3 *Reproduction of Figure 1, Case 3, of Hsieh et al. (2009)*

4. Exploring Average Total Inventory Cost: An In-depth Analysis

For inventory models with two trade credit financing, M is the retailer's trade credit period offered by the supplier and N is the customer's trade credit period offered by the retailer, with the condition $N \le M$.

The computation of interest earned pertains to items sold prior to M, a scenario that can be divided into two situations: occurrences preceding N, and those after N.

Concerning items sold before , the retailer doesn't receive payment until , resulting in accrued interest attributable to the cumulative demand during (denoted as). This accumulated interest is generated over the duration , giving rise to an accrued interest denoted as:

Concerning items sold before N, the retailer doesn't receive payment until N, resulting in accrued interest attributable to the cumulative demand during$[0, N]$, denoted as $\int_0^{min\{N,T\}} f(t)dt$ that will produce interest for the period $[N, M]$ such that the accumulated interest is denoted as

$$\int_0^{min\{T,N\}} (M-N)f(t)dt. \quad (4\text{-}1)$$

For those items sold after$N$ and before M, say demand $f(t)$ at the time t, the retailer doesn't settle the invoice immediately at t. Instead, the payment will be paid at *M*. When the demand $f(t)$ occurs, so the interest will be generated during $[t, M]$ to imply the accumulated interest is expressed as

$$\int_{\min\{T,N\}}^{\min\{T,M\}} (M-t) f(t) dt \quad (4\text{-}2)$$

Based on our above discussion of Eqs. (4-1) and (4-2), we will compute the interest earned for three different Cases: Case 1, $M \le T$; Case 2, $N \le T \le M$ and Case 3, $0 < T \le N$.

For Case 1 with $M \le T$, by Eqs. (4-1) and (4-2), we derive the total interest earned as

$$(M-N)\int_0^N f(t)dt + \int_N^M (M-t)f(t)dt \tag{4-3}$$

If we compare Eq. (4-3) with Eq. (3-1) to reveal that Hsieh et al. (2009) overlooked the interest earned for those items sold during $[0, N]$.

Based on our above discussion, we summarize our results in the next equation.

For Case 1,with $M \le T$, the average cost $AC_1(T)$ is revised as

$$\begin{aligned} AC_1(T) = \frac{1}{T}\Big\{ & A + c\int_0^T \left[e^{g(t)} - 1\right] f(t)dt + h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)dudt \\ & + cI_k \int_M^T e^{-g(t)} \int_t^T e^{g(u)} f(u)dudt - pI_e \left(\int_N^M (M-t)f(t)dt + (M-N)\int_0^N f(t)dt \right) \Big\} \end{aligned} \tag{4-4}$$

For Case 2 with $N \le T \le M$, by Eqs. (4-1) and (4-2), we find the total interest earned as

$$\cdot \quad (M-N)\int_0^N f(t)dt + \int_N^T (M-t)f(t)dt \tag{4-5}$$

If we compare the interest earned in Eqs. (3-2) and (4-5), so we rewrite those in Eq. (3-2) as

$$\begin{aligned} & \int_N^T (T-t)f(t)dt + (M-T)\int_0^T f(t)dt \\ = & \int_N^T (T-t)f(t)dt + (M-T)\int_0^N f(t)dt + (M-T)\int_N^T f(t)dt \\ = & \int_N^T (M-t)f(t)dt + (M-T)\int_0^N f(t)dt \end{aligned} \tag{4-6}$$

Now we compare Eqs. (4-5) and (4-6) to find that Hsieh et al. (2009) neglected the following term,

$$(T-N)\int_0^N f(t)dt \tag{4-7}$$

that is the items sold during $[0, N]$, the interest earned during the interval $[N, T]$ was overlooked by Hsieh et al. (2009).

Based on the above discussion, we obtain the following result.

For Case 2 with $N \le T \le M$, the average cost $AC_2(T)$ is revised as

$$\begin{aligned} AC_2(T) = \frac{1}{T}\Big\{ & A + c\int_0^T \left[e^{g(t)} - 1\right] f(t)dt + h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)dudt \\ & - pI_e \left[(M-N)\int_0^N f(t)dt + \int_N^T (M-t)f(t)dt \right] \Big\}. \end{aligned} \tag{4-8}$$

For Case 3 with $0 < T \le N \le M$, by Eqs. (4-4) and (4-2), we locate the total interest earned as

$$(M-N)\int_0^T f(t)dt \tag{4-9}$$

If we compare Eq. (4-9) with Eq. (3-3) to know that Hsieh et al. (2009) and this paper, both derive the identical result. Hence, we accept the findings of $AC_3(T)$ proposed by Hsieh et al. (2009).

5. Exploring First Derivatives of Their Objective Functions: A Detailed Analysis

From our result of Eq. (4-4), we derive our revised findings,

$$\frac{dAC_1(T)}{dT}=\frac{-A}{T^2}+\frac{c\left[Tf(T)(e^{g(T)}-1)-\int_0^T(e^{g(T)}-1)f(t)dt\right]}{T^2}$$
$$+\frac{h\left[Tf(T)\int_0^T e^{g(T)-g(t)}dt-\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)dudt\right]}{T^2}$$
$$+\frac{cI_kTf(T)\int_M^T e^{g(T)-g(t)}dt-cI_k\int_M^T e^{-g(t)}\int_t^T e^{g(u)}f(u)dudt}{T^2}$$
$$+\frac{pI_e\int_N^M(M-t)f(t)dt}{T^2}+\frac{pI_e(M-N)\int_0^N f(t)dt}{T^2} \tag{5-1}$$

The last term of Eq. (5-1) is an extra result proposed by us.

Moreover, based on our revised results of Eq. (4-8) for $AC_2(T)$, we clearly express our results for $dAC_2(T)/dT$ in the following,

$$\frac{dAC_2(T)}{dT}=\frac{-A}{T^2}+\frac{c\left[Tf(T)(e^{g(T)}-1)-\int_0^T(e^{g(t)}-1)f(t)dt\right]}{T^2}$$
$$+\frac{h\left[Tf(T)\int_0^T e^{g(T)-g(t)}dt-\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)dudt\right]}{T^2}$$
$$+\frac{pI_e}{T^2}\left[\int_N^T(T-t)f(t)dt+(M-T)\int_0^T f(t)dt\right]\Bigg\}$$
$$-\frac{pI_e}{T}\left[\int_N^T f(t)dt-\int_0^T f(t)dt+(M-T)f(T)\right]\Bigg\}-\frac{pI_eN}{T^2}\int_0^N f(t)dt \tag{5-2}$$

Based on our amendment, from Eq. (5-2), we derive that

$$\lim_{T\to M^-}AC_2'(T)=\frac{pI_e}{M^2}\int_N^M(M-t)f(t)dt-\frac{pI_e}{M}\left[\int_N^M f(t)dt-\int_0^M f(t)dt\right]-\frac{pI_eN}{M^2}\int_0^N f(t)dt \tag{5-3}$$

We can simplify Eq. (5-3) as

$$\lim_{T\to M^-}AC_2'(T)=\frac{pI_e}{M^2}\int_N^M(-t)f(t)dt+\frac{pI_e}{M}\left(\int_0^N f(t)dt+\int_N^M f(t)dt\right)-\frac{pI_eN}{M^2}\int_0^N f(t)dt \tag{5-4}$$

and then we obtain that

$$\lim_{T\to M^-}AC_2'(T)=\frac{pI_e}{M^2}\int_N^M(M-t)f(t)dt+\frac{pI_e(M-N)}{M^2}\int_0^N f(t)dt \tag{5-5}$$

On the other hand, based on Eq. (5-1), we find that

$$\lim_{T\to M^+}AC_1'(T)=\frac{pI_e}{M^2}\int_N^M(M-t)f(t)dt+\frac{pI_e(M-N)}{M^2}\int_0^N f(t)dt \tag{5-6}$$

Now, we compare Eqs. (5-5) and (5-6), then the desired finding

$$\lim_{T\to M^+} AC_1'(T) = \lim_{T\to M^-} AC_2'(T) \tag{5-7}$$

is achieved that is convincing support for our revisions.

We directly quote the result of Hsieh et al. (2009) for $AC_3'(T)$, then

$$\frac{dAC_3(T)}{dT} = \frac{-A}{T^2} + \frac{c\left[Tf(T)(e^{g(T)}-1) - \int_0^T (e^{g(t)}-1)f(t)dt\right]}{T^2}$$
$$+\frac{h\left[Tf(T)\int_0^T e^{g(T)-g(t)}dt - \int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)dudt\right]}{T^2}$$
$$-\frac{pI_e(M-N)}{T^2}\left[Tf(T) - \int_0^T f(t)dt\right] \tag{5-8}$$

According to our revisions, from Eq. (5-2), we know that

$$\lim_{T\to N^+} AC_2'(T) = \frac{pI_e}{N^2}(M-N)\int_0^N f(t)dt$$
$$-\frac{pI_e}{N}\left[-\int_0^N f(t)dt + (M-N)f(N)\right] - \frac{pI_eN}{N^2}\int_0^N f(t)dt \tag{5-9}$$

We can simplify Eq. (5-9) as

$$\lim_{T\to N^+} AC_2'(T) = -\frac{pI_e(M-N)}{N^2}\left[Nf(N) - \int_0^N f(t)dt\right] \tag{5-10}$$

Using Eq. (5-8), we obtain that

$$\lim_{T\to N^-} AC_3'(T) = -\frac{pI_e(M-N)}{N^2}\left[Nf(N) - \int_0^N f(t)dt\right] \tag{5-11}$$

Next, we compare Eqs. (5-10) and (5-11), then the desired finding

$$\lim_{T\to N^+} AC_2'(T) = \lim_{T\to N^-} AC_3'(T) \tag{5-12}$$

is achieved that will be another strong support for our amendments.

6. Analyzing the Smoothly Connected Property of Their Objective Functions: In-depth Examination

We have derived an improved version for $AC_1(T)$ as Eq. (4-4) and for $AC_2(T)$ as Eq. (4-8). We accept$AC_3(T)$ derived by Hsieh et al. (2009). Based on our revised version for $AC_1(T)$and $AC_2(T)$, in Section 5, we already prove that the first derivatives having the desired property of Eqs. (5-13) and (5-20) to show that this inventory model has the *smoothly connected property* mentioned by Lin et al. (2012). In the following, we cite from Lin et al. (2012) for the formal definition of ―*smoothly connected* ", "For two functions, say $g(x)$ for $a_1 \le x \le a_2$ and $h(x)$ for $a_2 \le x \le a_3$, they are smoothly connected if and only if

$$g(a_2) = h(a_2), \tag{6-1}$$

and

$$\lim_{x \to a_2^-} \frac{d}{dx} g(x) = \lim_{x \to a_2^+} \frac{d}{dx} h(x) \tag{6-2}$$

Based on Eq. (6-1), we begin to check whether or not $AC_2(M) = AC_1(M)$?

First, we recall the results of Hsieh et al. (2009). We using Eqs. (3-1) and (3-2), then Hsieh et al. (2009) derived that

$$AC_1(M) = \frac{1}{M}\left\{A + c\int_0^M \left(e^{g(t)} - 1\right) f(t)\,dt\right.$$
$$\left. h\int_0^M e^{-g(t)} \int_t^M e^{g(u)} f(u) du dt - pI_e \int_N^M (M-t) f(t) dt\right\}$$
$$= AC_2(M) \tag{6-3}$$

Based on Eqs. (4-4) and (4-8), then we obtain that

$$AC_1(M) = \frac{1}{M}\left\{A + c\int_0^M \left(e^{g(t)} - 1\right) f(t)\,dt\right.$$
$$\left. h\int_0^M e^{-g(t)} \int_t^M e^{g(u)} f(u) du dt - pI_e \left[(M-N)\int_0^N f(t)\,dt + \int_N^M (M-t) f(t) dt\right]\right\}$$
$$= AC_2(M) \tag{6-4}$$

We compare Eqs. (6-3) and (6-4) to point out that Hsieh et al. (2009) derived questionable $AC_1(T)$ and $AC_2(T)$. Accidently, their questionable results also imply $AC_1(M) = AC_2(M)$ in a false representation.

Similarly, based on Eq. (6-1), we begin to check whether or not $AC_2(N) = AC_3(N)$?

First, we recall the results of Hsieh et al. (2009). We usingEqs. (3-2) and (3-3), then Hsieh et al. (2009) derived that

$$AC_2(N) = \frac{1}{N}\left\{A + c\int_0^N \left(e^{g(t)} - 1\right) f(t)\,dt + h\int_0^N e^{-g(t)} \int_t^N e^{g(u)} f(u) du dt\right.$$
$$\left. -pI_e \left[(M-N)\int_0^N f(t)\,dt\right]\right\}$$
$$= AC_3(N). \tag{6-5}$$

We apply Eq. (4-8) for $AC_2(T)$ and Eq. (3-3) for $AC_3(T)$, then the identical result as Eq. (6-5) is obtained.

From Eqs. (5-13) and (6-4), we know that $AC_1(T)$ and $AC_2(T)$ are smoothly connected. Using Eqs. (5-20) and (6-5), we know that $AC_3(T)$ and $AC_2(T)$ are smoothly connected.

Therefore, we show that the three objective functions for the inventory model proposed by Hsieh et al. (2009) have smoothly connected property.

7. Examining the Second Derivatives in Hsieh et al.'s (2009) Analysis

We provide a brief discussion proposed by Hsieh et al. (2009) for the second derivatives for $\mathrm{AC}_1(\mathrm{T})$, $\mathrm{AC}_2(\mathrm{T})$ and $\mathrm{AC}_3(\mathrm{T})$, respectively.

Hsieh et al. (2009) obtained $d^2AC_1(T)/dT^2$, $d^2AC_2(T)/dT^2$, and $d^2AC_3(T)/dT^2$. However, in this paper, we will not cite their results because of their first derivatives for $dAC_1(T)/dT$, and $dAC_2(T)/dT$ of Eqs. (5-1) and (5-3) contained questionable results that had been pointed out by Eqs. (5-2) and (5-6). Moreover, in the following, we will show that deriving the second derivatives is unnecessary. Hence, to cite their questionable second derivatives is useless.

Without explicitly referring to their second derivatives, we still cite their three Propositions concerning their optimal solution.

Proposition 1 of Hsieh et al. (2009). If

$$2A > max\{(pI_e - cI_k)M^2f(M), pI_e(N^2f(M) + M^3f'(M))\} \tag{7-1}$$

$dAC_1(T)/dT$is a strictly increasing function of T and there exists a unique real solution $T^* \in [M, \infty)$ such that $AC_1(T)$ is minimum.

Proposition 2 of Hsieh et al. (2009). If

$$2A > max\{(pI_e - cI_k)M^2f(M), pI_e(N^2f(M) + M^3f'(M))\},$$

$dAC_2(T)/dT$is a strictly increasing function of T and there exists a unique real solution $T^* \in [N, M)$ such that $AC_2(T)$ is minimum.

Proposition 3 of Hsieh et al. (2009). If

$$2A > max\{(pI_e - cI_k)M^2f(M), pI_e(N^2f(M) + M^3f'(M))\},$$

$dAC_3(T)/dT$is a strictly increasing function of T and there exists a unique real solution $T^* \in (0, N)$ such that $AC_3(T)$ is minimum.

For the proof of their Proposition 1, the second derivatives, $d^2AC_1(T)/dT^2$, is used to prove $d^2AC_1(T)/dT^2 > 0$ with three auxiliary functions. The extra condition of Eq. (7-1) is proposed by Hsieh et al. (2009) to ensure the positivity of their three auxiliary functions.

In the following, we begin our discussion to show that the existence of the optimal solution without the extra condition of Eq. (7-1). Moreover, the derivations of the second derivatives are unnecessary.

We begin to derive our theoretical results for $AC_1(T)$, $AC_2(T)$, and $AC_3(T)$, without the extra condition of Eq. (7-1) proposed by Hsieh et al. (2009).

Theorem 1.

There exists a unique real solution $T^* \in [M, \infty)$ such that $AC_1(T)$ is minimum.

(Proof)

Based on Eq. (4-4), we observe that in the numerator, the constant term, *A* and the term with the coefficient pI_e does not contain the variable *T*, such that

$$\lim_{T\to\infty} \frac{A - pI_e\left(\int_N^M (M-t)f(t)dt + (M-N)\int_0^N f(t)dt\right)}{T} = 0 \quad (7\text{-}2)$$

We recall that $g(t) = \int_0^t \theta(u)\,du$ where $\theta(t)$ is the deterioration function with $\theta'(t) \geq 0$ and $0 < \theta(t) \leq 1$ so $\theta(t)$ is a non-decreasing function. We can pick a point, say ε with $\theta(\varepsilon) > 0$ and then

$$g(t) \geq \int_\varepsilon^t \varepsilon\,du = \varepsilon(t-\varepsilon), \quad (7\text{-}3)$$

for $t > \varepsilon$.

$f(t)$ is the demand rate with $f'(t) > 0$ so $f(t)$ is an increasing function to imply that

$$f(t) > f(\varepsilon), \quad (7\text{-}4)$$

for $t > \varepsilon$.

We recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, to imply that

$$e^x - 1 > x, \quad (7\text{-}5)$$

for $x > 0$.

Using Eqs. (7-3) and (7-5), and exponential function is an increasing function, we obtain that

$$e^{g(t)} - 1 > \varepsilon(t-\varepsilon) \quad (7\text{-}6)$$

After the above preparation, we estimate that

$$\int_0^T \left(e^{g(t)} - 1\right)f(t)dt > \int_\varepsilon^T \varepsilon(t-\varepsilon)f(\varepsilon)dt = \frac{\varepsilon}{2}f(\varepsilon)(T-\varepsilon)^2. \quad (7\text{-}7)$$

The other two terms with coefficients *h* and cI_k, are both positive such that using Eqs. (7-2) and (7-7), we derive that

$$AC_1(T) \geq \frac{\int_0^T \left(e^{g(t)} - 1\right)f(t)dt}{T} > \frac{\varepsilon f(\varepsilon)(T-\varepsilon)^2}{2T} \quad (7\text{-}8)$$

Based on Eq. (7-8), we show that

$$\lim_{T\to\infty} AC_1(T) = \infty \quad (7\text{-}9)$$

Using Eq. (7-9), there is a point, say Q, such that

$$AC_1(T) \geq AC_1(M) \tag{7-10}$$

for $T > Q$.

Owing to Eq. (7-10), we can change the domain for the minimum problem with $AC_1(T)$ from $[M, \infty)$ to $[M, Q]$ which is a compact set.

After we reduce our domain for $AC_1(T)$ to a compact set, then the minimum problem will be achieved within $[M, Q]$.

Theorem 2.

There exists a unique real solution $T^* \in [N, M]$ such that $AC_2(T)$ is minimum.

(Proof)

Based on Eq. (4-8), $AC_2(T)$ is a continuous function defined on a compact domain, $[N, M]$ such that the minimum value of $AC_2(T)$ is attained within $[N, M]$.

Theorem 3.

There exists a unique real solution $T^* \in (0, N]$ such that $AC_3(T)$ is minimum.

(Proof)

Based on Eq. (3-3), the numerator of $AC_3(T)$ contains three integrations concerning T and a constant term, A. When $T \to 0$, those three integrations approach to zero such that the numerator is dominated by the constant term and then

$$\lim_{T \to 0} AC_3(T) = \infty. \tag{7-11}$$

Based on Eq. (7-11), given a fixed number, say $AC_3(N)$, there is a number, denoted as δ, such that if $0 < T < \delta$, then $AC_3(T) > AC_3(N)$.

Therefore, for the minimum problem, we can change the domain from$(0, N]$ to a compact subset $[\delta, N]$ and then the minimum value of $AC_3(T)$ is achieved within $[\delta, N]$.

From our Theorems 1, 2, and 3, three objective functions $AC_1(T)$, $AC_2(T)$, and $AC_3(T)$, will attain their minimums within their domain, $[M, \infty)$, $[N, M]$, and $(0, N]$, respectively.

At last, we compare these three local minimums to derive the global minimum.

8. Numerical Examples

Owing to the objective functions of Case 1 and Case 2 are revised and the first derivatives for Case 1 and Case 2 are also improved, we will rerun their numerical examples 1 and 2 in Hsieh et al. (2009).

Example 1. We follow Hsieh et al. (2009) to assume that $A = 200$, $c = 30$, $h = 6$, $p = 50$, $I_k = 0.15$, $I_e = 0.12$, $\theta(t) = \alpha\beta t^{\beta-1}$, with the scale parameter $\alpha = 0.08$, and the shape parameter $\beta = 1.5$, $M = 45/365$, $N = 15/365$, and $f(t) = 1000 + 100t + 20t^2$. We list our findings in the following table.

Table 1 *Local Optimal Solutions of Example 1*

	i = 1	i=2	i=3
T_i^*	0.1834	0.1233	0.0289
$AC_i(T_i^*)$	1519.2967	1691.4334	4501.1563

We observe Table 1. to find that $AC(T^*) = AC_1(T_1^*)$ with the optimal value of $T^* = T_1^* =$ 0.1834. and the optimal value of $AC(T)$ is $AC_1(T_1^*)$ =1519.2967.

For completeness, we recall the finding of Hsieh et al. (2009) with their minimum point $T^* =$ 0.1885 and then we compute $AC_1(0.1885) = 1520.14$ to illustrate the effectiveness of our improvements.

Example 2. We still follow Hsieh et al. (2009) to change the parameter of *M* and *N* as $M = 60/365$, and $N = 30/365$, the rest parameters are not altered. We list our results in the next table.

Table 2 *Local Optimal Solutions of Example 2*

	i = 1	i=2	i=3
T_i^*	0.1864	0.1644	0.0822
$AC_i(T_i^*)$	1366.7596	1384.1699	2200.4047

From Table 2 it can be found that the optimal value of $T^* = T_1^* =$ 0.1864 and the optimal value of $AC(T)$ is $AC_1(T_1^*)$ =1366.7596.

For completeness, we refer to the result of Hsieh et al. (2009) with their minimum point $T^* =$ 0.1638 and then we compute $AC_1(0.1638) = 1385.17$ to demonstrate the usefulness of our revisions.

In Examples 1 and 2, the global minimum as T_3^* occurs in the domain $[M, \infty)$. With the help of an anonymous reviewer, we point out that studying the influence of parameters to construct numerical examples with T_1^* in $(0, N)$ or T_2^* in (N, M) will help researchers realize inventory models proposed by Hsieh et al. (2009).

Moreover, Proposition 4 of Hsieh et al. (2009), which derived several cases, directly decided the location of the global minimum. To examine Proposition 4 will be an interesting research topic in the future.

9. Conclusion

In this study, we have undertaken a comprehensive analysis of the influential work presented by Hsieh et al. (2009), which introduces an inventory model incorporating dual trade credit policies. Our investigation has illuminated certain questionable aspects within their objective functions, with potential ramifications on the validity of their first derivative computations. The possibility arises that their established minimum solution might not necessarily correspond to the optimal solution.

To address these concerns, we have embarked on a journey of refinement. We have commenced by enhancing their objective functions pertaining to $AC_1(T)$ and $AC_2(T)$, which has paved the way for a meticulous reevaluation of their results concerning $dAC_1(T)/dT$ and

$dAC_2(T)/dT$. This concerted effort has led us to establish the smoothly connected property inherent to this inventory model.

Intriguingly, our advancements dispense with the need for their supplementary condition, as we substantiate that their three sub-objective functions inherently possess minima within their respective domains. By achieving this, our findings substantially reinforce the theoretical underpinnings of the proposed inventory model, serving as a robust framework that rectifies and enhances the existing theoretical proof. This paper significantly contributes to the refinement and validation of the influential work put forth by Hsieh et al. (2009).

Acknowledgements

The authors express their gratitude to an anonymous reviewer for revising our Abstract. We also show our appreciation to another reviewer for improving the quality of our manuscript.

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(Received October 2023; accepted February 2024)