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An Optimal Production Policy of Stochastic Inventory System with and without Lost Sales

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Abstract

We consider Markov decision model of production-inventory system with defective and deteriorating items. Two cases of demand; random variable and price dependent are discussed. Our mathematical model discusses four strategies of production, and pricing strategy. Production strategies includes with and without losing sales, and the mixed strategy. Our aim is to determine the optimal policy of production and pricing strategy, thus maximizes the expected average profit. Also, our model takes into account the effect of defective and deteriorating products in the actual inventory. Numerical application and sensitivity analysis are conducted. The results show a strategy without losing sales is the best compared with other strategies. Meanwhile, other strategies can apply with limited production and capital. The sell price is 250% of the unit cost leads to high profit compared with other prices.

Keywords: Production-inventory system, Markov decision model, defective and deteriorating products, stochastic demand, losing sales.

1. Introduction

An optimal production policy in one of the important factors of inventory system that discussed in recent decades. Many factors affect on the system, such as demand, raw materials, defective and deteriorating products. Practical, machine age and quality of raw materials lead to some defective products. Also, product lifetime has an effect on the real storage quantity due to some deterioration during storage period. The production policy should make a balance between satisfying demand and lost sales to maximize the profit.

Many researchers have developed mathematical models of inventory system with stochastic demand. The Markov decision process (MDP) was wide discussed the inventory system with stochastic demand. Yin et al. [33] determined the replenishment amount of two distributions of the demand; normal and Exponential. They discussed two costs of the inventory system; manufacturing and shortage. Jin et al. [13] investigated assembly system with returned products according to a compound Poisson process. They classified remanufacturing products into many types, according to the quality. Multi-stage of production, limited production of every stage, and random lead time was investigated by Vercraene and Gayon [31]. They discussed two options of return products; accept and reject to determine the optimal level of base-stock. The same demand assumption of Jin et al. [13], Ahiska and Kurtul [1] developed a hybrid manufacturing system with stochastic returned products. They considered the sale price of remanufacturing is lower than manufacturing product price. Determine the optimal order quantity of the milk powder with l cost of order, holding and shortage in the supermarket, was studied by Mubiru [20]. Feng et al. [10] studied system with many product, discount rate and neglect lead time. The aim is to determine the replenishment policy with minimized the total cost.

Several researchers have developed other models of inventory system with stochastic demand. Kim and Jeong [14] discussed buyer-supplier model with a single item in the case of periodic review policy to determine the optimal cycle. Multistage inventory system with setup cost, fixed lead time, continuous review policy and replenishment policy (r, Q), was described by Hu and Yang [12].

Remanufacturing model with uncertain reprocessing amount and uncertain supply, was discussed by Wen et al. [32] and Li et al. [17], respectively. The first one minimized the total cost of the model with two-stage, multi-period and compensation function. Meanwhile, the second one considered two strategies of remanufacturing and pricing. Also, with remanufacturing system and uncertain, Fang, et al. [8] determined the optimal strategy of operation. The aim is to minimize the total cost of holding, production, remanufacturing, disposal and shortage. Two models of inventory system; optimal control and quadratic programming, with deterioration and limited inventory, was discussed by Dhaiban [6]. More recently, Assid et al, [2] developed a control policy of production and remanufacturing to determine the storage size of returned and new products.

A defective product is one of the important factors in the inventory system that discussed by several researchers. Numbers of shipment for each production cycle, retail price and lot size of the supplier-retailer model, was discussed by Soni and Patel [27]. EPQ model discussed by Farsijani et al. [9] and Sarkar et al. [25]. Farsijani et al. [9] determined the optimal quantity of order and shipments of the system with many products and repair defective products. They have taken into account and constant rate of repairing defective items. Meanwhile, Sarkar et al. [25] compared three models of inventory system with three defective rates. Three different distributions of the defective rate in the case of single stage and backorders. Three rates of random defective also studied by Priyan and Uthayakumar [24]. They established an inventory model to find the optimal number of shipments from the vendor to the buyer, as well as lot size and setup cost. Reduce the defective products with demand dependent price and investment depends on quality improvement was studied by Datta [5]. Maintenance policy with defective products was investigated by Nobil and Sedigh [21] and Bouslah et al. [4]. Nobil and Sedigh [21] determined the economic batch quantity and the optimal policy of maintenance. A model with two-machine line and two choices of defective products; accept and reject was addressed by Bouslah et al. [4]. More recently, Tayyab et al. [29] addressed fuzzy demand, multi-stage production, single product, rework of defective products to. The aim is to minimize the total cost of inspection, holding and order.

Several researchers have addressed an inventory system with stochastic demand and defective items. Two cases of defective products, and random demand that adhere to normal and Exponential distribution, were addressed by Liu et al. [18] and Bhowmick and Samanta [3], respectively. Kutzner and Kiesmller [15] studied periodic review policy to minimize the total cost of inspection and backorder. MDP model of inventory system with shortage and two strategies of disposal the defective items were discussed by Dhaiban and Aziz [7].

Products are subject to deteriorate during storage due to several reasons, such as lifetime. Thus, the actual inventory that satisfies the demand will decrease. An inventory system with deteriorating and defective products was addressed by a few researchers. Many plant of production, single plant of reworking and compared two models was addressed by Tai [28]. Yu [34] discussed EOQ model with delay in payment and compensation policy. Meanwhile, Lee and Kim [16] discussed the direct sell strategy of defective items without storage in the distribution model. Screening process conduct by retailer was discussed by Moussawi-haidar et al. [19], with backorder and discount selling price. Uthayakumar and Tharani [30] developed rework model with $(n, 1)$ policy, one cycle for production setup, rework setup, and two types of demand. Several studies have discussed inventory system with only deteriorating products. Weibull deterioration rate, shortage, and deterministic demand, were investigated by Pervin et al. [23] and Sharma et al [26] to determine the optimal replenishment policy and economic order quantity, respectively. More recently, Patel [22] determined the best pricing strategy that maximize the profit of the single product model.

The contribution of this paper is to formulate a Markov decision model of the production - inventory system. Inventory system with defective, deterioration, and stochastic demand. Moreover, the demand depends on the sell price and in the same time random variable. The model determines the pricing strategy and the optimal production policy. Four production policies; with losing sales, without losing sales and mixed strategy. The production policies and pricing strategy are suitable with many practical situations. The production capacity, capital, raw material, storage and so on, are effective factors on the decision making. Sensitivity analysis was shown the effect of holding cost and losing sales cost on the production policy. Table 1 shows our contribution compared other literature.

2. Production-Inventory System

2.1. Problem description

A production-inventory system in this paper can described as follows:

1. The company produces one product, and checking process happens direct for all product units.
2. A specific percent of production is defective.

Table 1: Summary of literature review.

Author(s)	Defective	Deterioration	Random Demand	Lost sales (with & without)	MDP	Sensitivity Analysis	Comparison (several models)	Periodic review policy	Demand is Random & Price Dependent
Yin et al. [23]			√	√	√		√	√	
Jin et al. [13]			√	√	√			√	
Vercraene and Gayon [31]			√	√	√		√	√	
Mubiru [20]			√	√	√			√	
Ahiska and Kurtul [1]			√	√	√			√	
Feng et al. [10]			√	√	√	√	√		
Kim and Jeong [14]			√	√				√	
Hu and Yang [12]			√	√		√			
Wen et al. [32]			√						
Li et al. [17]			√	√		√	√		
Fang et al. [8]			√	√		√			
Dhaiban [6]		√	√			√	√	√	
Assid et al. [2]			√	√			√		
Farsijani et al. [9]	√				√			√	
Soni and Patel [27]	√						√		
Sarker et al. [25]	√					√	√	√	
Priyan and Uthayakumar [24]	√					√	√	√	
Datta [5]	√			√		√			
Nobil and Sedigh [21]	√								
Bouslah et al. [4]	√					√			
Tayyab et al. [29]	√					√			
Liu et al. [18]	√		√	√			√		
Bhowmick and Samanta [3]	√		√	√		√	√		
Kutzner and Kiesmller [15]	√		√				√		
Dhaiban and Aziz [7]	√		√	√	√			√	
Tai [28]	√	√		√			√		
Yu [34]	√	√		√		√			
Lee and Kim [16]	√	√				√			
Moussawi-Haidar et al. [19]	√	√		√		√			
Uthayakumar and Tharani [30]	√	√				√	√		
Pervin et al. [23]		√		√		√			
Sharma et al. [26]		√		√		√	√		
Patel [22]		√		√		√			
This article	√	√	√	√	√	√	√	√	√

3. Products during storage are subject to deterioration by specific percent. Assumption of deterioration by specific percent of inventory, due to the storage period does not exceed one month.
4. Defective and deteriorating products dispose direct.
5. Two cases of demand; random variable and pricing dependent.
6. The production rate depends on the inventory level of the previous month with some deteriorating products. Also depends on the demand of the current month, and the defect percent.
7. The unit cost includes production, holding, disposal, check, and lost sales.

2.2. Notations

The following variables and parameters are used:

I_t : The inventory level at time t.

I_0 : The initial inventory level at the beginning of the planning period.

N_t : The production rate that satisfies the demand.

D_t : The stochastic demand.

UD_t : The upper limit of the demand.

LD_t : The lower limit of the demand.

δ : The percentage of defective products.

ϑ : The percentage of deteriorated products.

R_t : The production cost.

K_t : The cost of the checking production.

H_t : The holding cost.

L_t : The disposing cost of the defective and deteriorated products.

G_t : The lost sales cost.

A_t : Sales.

O : The expected average profit.

3. Mathematical Model

3.1. Markov decision processes (MDP)

Markovian property to any stochastic process I_t is as follows (Hillier and Lieberman [11]):

$$P\{I_{i+1} = j \mid I_0 = k_0, I_1 = k_1, \ldots,\ I_t = i\} = P\{I_{i+1} = j \mid I_t = i\}. \tag{3.1}$$

According to the Markov property, the happen probability of future event depends on present state, which means past events do not have any effect. Transition probabilities in the Markov chain are as follows:

$$P\{I_{t+1} = j \mid I_t = i\} = P\{I_1 = j \mid I_0 = i\} \quad \text{for all } i, j, t. \tag{3.2}$$

Eq. (3.2) represents transition probability for one-step, transition probabilities for n steps is as follows:

$$\begin{aligned} P\{I_{t+n} = j \mid I_t = i\} &= P\{I_n = j \mid I_0 = i\} \quad \text{for all } i, j, t, \\ P_{ij}^{(n)} &= P\{I_{t+n} = j \mid I_t = i\}. \end{aligned} \tag{3.3}$$

Transition probability must be greater than or equal to zero, and sum of transition probabilities for any state must be equal to one. If transition probabilities stay constant without change over time that means stationary transition probabilities are as follows:

$$\begin{aligned} p_{00} = p_{10} = p_{20} = \cdots = p_{i0}, \\ p_{01} = p_{11} = p_{21} = \cdots = p_{i1}, \\ \vdots \\ p_{0j} = p_{1j} = p_{2j} = \cdots = p_{ij}. \end{aligned} \tag{3.4}$$

The steady state probability γ_j represents a probability after many transitions, which is becoming independent of the initial state probability.

$$\begin{aligned} &\lim_{n\to\infty} P_{ij}^{(n)} = \gamma_j, \\ &\gamma_j = \sum_{i=0}^{M} \gamma_i P_{ij} \quad , \quad j = 0, 1, 2, \ldots, M, \\ &\sum_{j=0}^{M} \gamma_j = 1. \end{aligned} \tag{3.5}$$

For example, three states of transition probabilities, γ_j can be found by solving the last three equations simultaneously:

$$\begin{aligned} \gamma_0 &= \gamma_0 p_{00} + \gamma_1 p_{10} + \gamma_2 p_{20}, \\ \gamma_1 &= \gamma_0 p_{01} + \gamma_1 p_{11} + \gamma_2 p_{21}, \\ \gamma_2 &= \gamma_0 p_{02} + \gamma_1 p_{12} + \gamma_2 p_{22}, \\ 1 &= \gamma_0 + \gamma_1 + \gamma_2. \end{aligned} \tag{3.6}$$

3.2. Transition probabilities

Inventory level can be described by Eq. (3.7):

$$I_{t+1} = I_t + N_{t+1} - D_{t+1}. \tag{3.7}$$

Inventory level represented by the inventory level at the end of the last month addition to production rate, that subtracting from the monthly demand. Eq. (3.7) represents the production-inventory system without defective and deteriorating items. Eq. (3.8) represents the effect of defect products on the system:

$$I_{t+1} = I_t + (1-\delta)N_{t+1} - D_{t+1}. \tag{3.8}$$

By inserting the effect of the deteriorating items, Eq. (3.8) becomes as follows:

$$I_{t+1} = (1-\vartheta)I_t + (1-\delta)N_{t+1} - D_{t+1}. \tag{3.9}$$

The production rate is described by the function of inventory. From Eq. (3.9), the integer production rate (Int.) can be formulated as follows:

$$\begin{aligned} (1-\delta)N_{t+1} &= D_{t+1} - (1-\vartheta)I_t, \\ INT.(N_{t+1}) &= \frac{D_{t+1} - (1-\vartheta)I_t}{(1-\delta)}. \end{aligned} \tag{3.10}$$

The production rate must take into account defective and deteriorating products to satisfy the demand. To determine the optimal production policy, we suggest four strategies are as follows:

1. The production rate that satisfies the upper limit of the demand ($S1$), which means without losing sales. Therefore, we can rewrite Eq. (3.10) as follows:

$$INT.(N_{t+1}) = \frac{UD_{t+1} - (1-\vartheta)I_t}{(1-\delta)}. \tag{3.11}$$

2. The production rate that satisfies 85% of the upper limit of the demand ($S2$), which means with a slight losing sales. Therefore, we can rewrite Eq. (3.10) as follows:

$$INT.(N_{t+1}) = \frac{0.85 * UD_{t+1} - (1-\vartheta)I_t}{(1-\delta)}. \tag{3.12}$$

3. The production rate that satisfies 75% of the upper limit of the demand ($S3$), which means with a significant losing sale. Therefore, we can rewrite Eq. (3.10) as follows:

$$INT.(N_{t+1}) = \frac{0.75 * UD_{t+1} - (1-\vartheta)I_t}{(1-\delta)}. \tag{3.13}$$

4. Mixed strategy ($S4$) that apply $S1$, $S2$ and $S3$ at the beginning, middle, and end of the planning period, respectively.

The demand is a random variable with a probability distribution. The losing sales happen when the demand exceeds the production and inventory.

$$D_{n+1} > (1-\vartheta)I_t + (1-\delta)N_{t+1}. \tag{3.14}$$

A transition probability matrix can be written as follows (three state):

$$P = \begin{array}{c c} \begin{array}{c} State \\ 0 \\ 1 \\ 2 \end{array} & \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix} \end{array} \end{array} \tag{3.15}$$

An inventory level is described as a probability level because of the stochastic demand. Therefore, Eq. (3.15) represents the inventory transition probabilities from one state to another. For example, p_{12} represents an inventory transition probability from state 1 to state 2.

4. Expected Average Profit of the System

Expected average profit can be found by subtracting the expected average cost from the expected average sales.

4.1. Expected average cost of the system

Suppose C_t represents the system cost in the state t. C_t is a random variable and the expected average cost is as follows:

$$E\Big[\frac{1}{n}\sum_{t=1}^{n} C_t\Big]. \tag{4.1}$$

The expected average cost of the system includes production, checking, holding, disposal and losing sales:

$$C = C(R_t, K_t, H_{t-1}, L_t, D_t). \tag{4.2}$$

Markov chain in the case of finite-state satisfy the following equation:

$$\lim_{n\to\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} p_{ij}^{(k)}\Big) = \gamma_j. \tag{4.3}$$

Then, the expected average cost can be written as follows:

$$\lim_{n\to\infty} E\Big[\frac{1}{n}\sum_{t=1}^{n} C(R_t, K_t, H_{t-1}, L_t, D_t)\Big] = \sum_{j=0}^{m} \mu_j \gamma_j \tag{4.4}$$

where $\mu_j = E[C(R_t, K_t, H_{t-1}, L_t, D_t)]$.

The losing sales cost happen when the demand exceeds the actual production and inventory:

$$\begin{aligned} p\{(D_{t+1} = {} & (1-\vartheta)I_t + (1-\delta)N_{t+1} + 1)\} \\ & + 2p\{D_{t+1} = (1-\vartheta)I_t + (1-\delta)N_{t+1} + 2\} \\ & + 3p\{D_{t+1} = (1-\vartheta)I_t + (1-\delta)N_{t+1} + 3\} + \cdots . \end{aligned} \tag{4.5}$$

4.2. Expected average sales of the system

The expected average sales can be written as follows:

$$\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{t=1}^{n} A_t\right] = \sum_{j=0}^{m} \psi_j \gamma_j \tag{4.6}$$

where $\psi_j = E(A_t)$.

From Eqs. (4.4) and (4.6), we can write the expected average profit as follows:

$$O = \sum_{j=0}^{m} \psi_j \gamma_j - \sum_{j=0}^{m} \mu_j \gamma_j. \tag{4.7}$$

5. Numerical Application and Sensitivity Analysis

5.1. Case 1: The stochastic demand

5.1.1. Numerical application

Consider an inventory system with the following parameter values:

- The production cost $r = 10\$$ per unit.
- The holding cost is equal to 20% from production cost, $h = 0.2 * 10 = 2\$$ per unit.
- The production checking cost is equal to 5% from production cost, $k = 0.05 * 10 = 0.5\$$ per unit.
- The sale price is equal to 300% from production cost, $a = 3 * 10 = 30\$$ per unit.
- The disposal cost is equal to 5% from the sale price, $l = 0.05 * 30 = 1.5\$$ per unit.
- The losing sales cost is equal to 25% from the sale price, $g = 0.25 * 30 = 7.5\$$ per unit.
- The defective percentage $\delta = 0.1$.
- The deterioration percentage $\vartheta = 0.2$.

The unit cost is 14$ without losing sales, and 21.5$ with losing sales. The demand is a random variable, according the following probability distribution (see Table 2):

According to the demand, we can divide inventory level to the six states as follows:

1. The First Strategy ($S1$)

The first strategy represents satisfy the upper limit of demand, which means without losing sales. From Table 2, the upper limit of demand is 100 units, so the production rate can be found from Eq. (3.11) as follows:

$$N(0) = \frac{100 - 0.8(0)}{0.9} = 112. \tag{5.1}$$

Table 2: The probability distribution of demand.

Demand	$P_r(D)$	$\sum P_r(D)$
50∼58	0.012	0.108
59∼72	0.014	0.196
73∼85	0.027	0.351
86∼94	0.025	0.225
95∼100	0.02	0.12

Table 3: States of the inventory level.

State	0	1	2	3	4	5
Inventory	0	1∼5	6∼14	15∼27	28∼41	42∼50

Eq. (5.1) represents the production rate is (112) units when the inventory level is zero without deteriorating units and (12) defective units. For state 1, the production rate is as follows:

$$N_l(1) = \frac{100 - 0.8(5)}{0.9} = 106; \qquad N_u(1) = \frac{100 - 0.8(1)}{0.9} = 110. \tag{5.2}$$

Eq. (5.2) represents the low production rate $N_l(1)$ is (106) units when the inventory level is (5) with one deteriorating unit and (10) defective units. The high production rate $N_u(1)$ is (110) units when the inventory level is (1) without deteriorating unit and (11) defective units. Table 4 shows the production rate, deteriorating, defective and actual inventory for all states:

Table 4: Inventory level and production rate ($S1$).

State	ϑI	$(1-\vartheta)I$	N_l	N_u	δN_l	δN_u	$(1-\delta)N_l$	$(1-\delta)N_u$
0	0	0	112	112	12	12	100	100
1	0∼1	1∼4	106	110	10	11	96	99
2	1∼3	5∼11	99	106	10	11	89	95
3	3∼5	12∼22	87	98	9	10	78	88
4	5∼8	23∼33	75	86	8	9	67	77
5	8∼10	34∼40	67	74	6	8	60	66

The transition matrix of inventory level depends on the demand, which means states represents inventory and figures inside the matrix represent the demand (see Tables 5 & 6).

Table 5: The Transition matrix of inventory level ($S1$).

State	0	1	2	3	4	5
0	100	95~99	86~94	73~85	59~72	50~58
1	100	95~99	86~94	73~85	59~72	50~58
2	100	95~99	86~94	73~85	59~72	50~58
3	100	95~99	86~94	73~85	59~72	50~58
4	100	95~99	86~94	73~85	59~72	50~58
5	100	95~99	86~94	73~85	59~72	50~58

Table 6: The transition probability matrix ($S1$).

State	0	1	2	3	4	5
0	0.02	0.1	0.225	0.351	0.196	0.108
1	0.02	0.1	0.225	0.351	0.196	0.108
2	0.02	0.1	0.225	0.351	0.196	0.108
3	0.02	0.1	0.225	0.351	0.196	0.108
4	0.02	0.1	0.225	0.351	0.196	0.108
5	0.02	0.1	0.225	0.351	0.196	0.108

Probabilities in Table 6 are found from the demand, such as $p_{01} = 0.1$ is equal to the total of demand probabilities from 95 to 99 units. The total of production and inventory (excluding defective and deteriorating products) must be equal to the upper limit of demand at any inventory state. Transition probabilities in the Table 6 represent the steady state probabilities that found by two iterations. Therefore, the steady state probability γ_j is as follows:

$$\gamma_0 = 0.02;\ \ \gamma_1 = 0.1;\ \ \gamma_2 = 0.225;\ \ \gamma_3 = 0.351;\ \ \gamma_4 = 0.196;\ \ \gamma_5 = 0.108.$$

From Eq. (4.2), we can find the expected average cost:

$$\mu_j = N * r + N * k + I * h + l * (\vartheta I + \delta N).$$

We take the integer average of inventory, production, deteriorating and defective for every state in Table 4, the expected average cost is:

$$\begin{aligned}\mu_0 &= 112(10) + 112(0.5) + 0(2) + 12(1.5) = 1194,\\ \mu_1 &= 109(10) + 109(0.5) + 3(2) + 13(1.5) = 1170.\end{aligned}$$

From Eq. (4.4), the expected average cost (monthly) is as follows:

$$\sum_{j=0}^{m} \mu_j \gamma_j = 0.02*1194+0.1*1170+0.255*1123+0.351*1039.5+0.196*944.5+0.108*861.5$$
$$= 1036.6\$.$$

From Eq. (4.6), the expected average sales (monthly) is as follows:

$$\psi_0 = A_0 * a = 100 * 30 = 3000$$
$$\psi_1 = A_1 * a = 97 * 30 = 2910$$
$$\psi_2 = 2700; \quad \psi_3 = 2370; \quad \psi_4 = 1980; \quad \psi_5 = 1620$$

$$\sum_{j=0}^{m} \psi_j \gamma_j = 0.02*3000+0.1*2910+0.255*2700+0.351*2370+0.196*1980+0.108*1620$$
$$= 2353.4\$.$$

From Eq. (4.7), the expected average profit is as follows:

$$O = 2353.4 - 1036.6 = 1316.8\$$$

2. The Second Strategy ($S2$)

The second strategy represents satisfy 85% of the upper limit of demand, which means with slight losing sales. The production rate can be found from Eqs. (3.11) and (3.12) as follows:

$$N(0) = \frac{85 - 0.8(0)}{0.9} = 95;$$
$$N_l(1) = \frac{85 - 0.8(5)}{0.9} = 90; \qquad N_u(1) = \frac{85 - 0.8(1)}{0.9} = 94.$$

From Eq. (3.11), the losing sales happen when the demand is greater than 85. Inventory states will become seven states after adding state 6 that represents the losing sales. Table 7 shows the production rate, deteriorating, defective and actual inventory for all states:

Table 7: Inventory level and production rate ($S2$).

State	ϑI	$(1-\vartheta)I$	N_l	N_u	δN_l	δN_u	$(1-\delta)N_l$	$(1-\delta)N_u$
0	0	0	95	95	10	10	85	85
1	0∼1	1∼4	90	94	9	10	81	84
2	1∼3	5∼11	82	89	8	9	74	80
3	3∼5	12∼22	70	81	7	8	63	73
4	5∼8	23∼33	58	69	6	7	52	62
5	8∼10	34∼40	50	57	5	6	45	51
6	0	< 0	95	95	10	10	85	85

Transition probabilities represent the steady state probabilities. That is due to the fact that the total of production and inventory (excluding defective and deteriorating products) must be equal 85, so Tables (8 & 9) include one row, which itself can be repeat to other rows (states).

Table 8: The Transition matrix of inventory level ($S2$).

State	0	1	2	3	4	5	6
0	85	80∼84	71∼79	58∼70	50∼57		>85

Table 9: The transition probability matrix ($S2$).

State	0	1	2	3	4	5	6
0	0.027	0.135	0.217	0.18	0.096		0.345

$\gamma_0 = 0.027; \quad \gamma_1 = 0.135; \quad \gamma_2 = 0.217; \quad \gamma_3 = 0.18; \quad \gamma_4 = 0.096; \quad \gamma_5 = 0; \quad \gamma_6 = 0.345.$

The empty cells in Tables 8 and 9 means cannot inventory move from state to another. Probabilities of state 6 are found by subtracting the total of probabilities in each row from one, for example:

$$p_{06} = 1 - (0.027 + 0.135 + 0.217 + 0.18 + 0.096) = 0.345.$$

From Eq. (4.4), we can find the expected average cost:

$$\begin{aligned}\mu_j &= N * r + N * k + I * h + l * (\vartheta I + \delta N)\\ &\quad + g\{P_r(D = 86) + 2P_r(D = 87) + \cdots + 15P_r(D = 100)\}\end{aligned}$$

$$\begin{aligned}\mu_0 &= 95(10) + 95(0.5) + 0(2) + 10(1.5) + 7.5\{0.025 + 2(0.025) + 3(0.025) + 4(0.025)\\ &\quad + 5(0.025) + 6(0.025) + 7(0.025) + 8(0.025) + 9(0.025) + 10(0.02) + 11(0.02)\\ &\quad + 12(0.02) + 13(0.02) + 14(0.02) + 15(0.02)\} = 1032.2\end{aligned}$$

$$\begin{aligned}\sum_{j=0}^{m} \mu_j \gamma_j &= 0.027{*}1032.2{+}0.135{*}1008.2{+}0.217{*}961.2{+}0.18{*}877.7{+}0.096{*}782.7\\ &\quad +0.345{*}1032.2 = 961.8\$.\end{aligned}$$

From Eq. (4.6), the expected average sale (monthly) is as follows:

$$\begin{aligned}\sum_{j=0}^{m} \psi_j \gamma_j &= 0.027{*}2550{+}0.135{*}2460{+}0.217{*}2250{+}0.18{*}1920{+}0.096{*}1620{+}0.345{*}2550\\ &= 2270.1\$\end{aligned}$$

From Eq. (4.7), the expected average profit is as follows:

$$O = 2270.1 - 961.8 = 1308.3\$.$$

3. The Third Strategy ($S3$)

The third strategy represents satisfy 75% of the upper limit of demand, which means with significant losing sales. The production rate can be found from Eqs. (4.11) & (4.13) as follows (see Table 10):

$$N(0) = \frac{75 - 0.8(0)}{0.9} = 84;$$
$$N_l(1) = \frac{75 - 0.8(5)}{0.9} = 79; \qquad N_u(1) = \frac{75 - 0.8(1)}{0.9} = 83.$$

Table 10: Inventory level and production rate ($S3$).

State	ϑI	$(1-\vartheta)I$	N_l	N_u	δN_l	δN_u	$(1-\delta)N_l$	$(1-\delta)N_u$
0	0	0	84	84	9	9	75	75
1	0∼1	1∼4	79	83	8	8	71	75
2	1∼3	5∼11	71	78	7	8	64	70
3	3∼5	12∼22	59	70	6	7	53	63
4	5∼8	23∼33	47	58	5	6	42	52
5	8∼10	34∼40	39	46	4	5	35	41
6	0	< 0	84	84	9	9	75	75

Tables (11 & 12) include one row, which itself can be repeat to other rows (states).

Table 11: The transition matrix of inventory level ($S3$).

State	0	1	2	3	4	5	6
0	75	70∼74	61∼69	50∼60			> 75

Table 12: The transition probability matrix ($S3$).

State	0	1	2	3	4	5	6
0	0.027	0.096	0.126	0.136			0.615

$$\gamma_0 = 0.027;\ \gamma_1 = 0.096;\ \gamma_2 = 0.126;\ \gamma_3 = 0.136;\ \gamma_4 = 0;\ \gamma_5 = 0;\ \gamma_6 = 0.615.$$

From Eq. (4.4), we can find the expected average cost:

$$\begin{aligned}\mu_j =& N*r + N*k + I*h + l*(\vartheta I + \delta N)\\ &+ g\{P_r(D=76) + 2P_r(D=77) + \cdots + 25P_r(D=100)\}\end{aligned}$$

$$\begin{aligned}\mu_0 =& 84(10) + 84(0.5) + 0(2) + 9(1.5) + 7.5\{0.027 + 2(0.027) + 3(0.027)\\ &+ 4(0.027) + 5(0.027) + 6(0.027) + 7(0.027) + 8(0.027) + 9(0.027) + 10(0.027)\\ &+ 11(0.025) + 12(0.025) + 13(0.025) + 14(0.025) + 15(0.025) + 16(0.025)\\ &+ 17(0.025) + 18(0.025) + 19(0.025) + 20(0.02) + 21(0.02) + 22(0.02)\\ &+ 23(0.02) + 24(0.02) + 25(0.02)\} = 952.2\end{aligned}$$

$$\begin{aligned}\sum_{j=0}^{m}\mu_j\gamma_j =& 0.027*1032.2+0.135*1008.2+0.217*961.2+0.18*877.7+0.096*782.7\\ &+0.345*1032.2 = 961.8\$.\end{aligned}$$

From Eq. (4.6), the expected average sale (monthly) is as follows:

$$\sum_{j=0}^{m}\mu_j\gamma_j = 0.027*952.2+0.096*926.7+0.126*881.2+0.136*797.7+0.615*952.2 = 919.8\$$$

From Eq. (4.6), the expected average sale (monthly) is as follows:

$$\sum_{j=0}^{m}\psi_j\gamma_j = 0.027*2250+0.096*2160+0.126*1950+0.136*1650+0.615*2250 = 2122\$$$

From Eq. (4.7), the expected average profit is as follows:

$$O = 2122 - 919.8 = 1202.2\$.$$

4. The Fourth Strategy ($S4$)

The fourth strategy represents a mixed strategy with applying all strategies. For example, planning period length is six months, so we apply $S1$ in the first two months, $S2$ at the third and fourth month, and $S3$ at the last two months.

The expected average cost of six months is as follows:

$$2*1036.6 + 2*961.8 + 2*919.8 = 5836.3\$.$$

The expected average sale of six months is as follows:

$$2*2353.4 + 2*2270.1 + 2*2122 = 13491\$.$$

The expected average profit of six months is as follows:

$$13491 - 5836.3 = 7654.7\$.$$

The expected average profit of six months for $S1$, $S2$ and $S3$ are 7900.9\$, 7849.9\$ and 7213.2\$, respectively.

The simulation results show the expected average cost of the third strategy ($S3$) is lower than other strategies. That is due to the fact a lower rate of production. Meanwhile, the expected average sales and profit of the first strategy ($S1$), without losing sales, is higher than other strategies. In general, $S1$ is the best strategy as shown in Figure 1 due to it satisfies the demand, thus increase the profit. The first strategy needs to unlimited available of capital, production rate, raw materials, good machines and so on. Meanwhile, using other strategies based on available amount of above-mentioned factors.

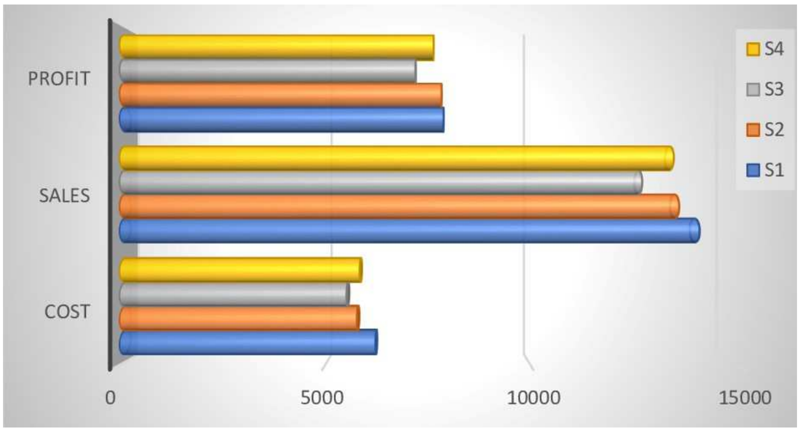

Figure 1: Expected averages of cost, sales and profit for all strategies.

5.1.2. Sensitivity analysis

In this section, we show the effect of changing the holding cost from 20% to 40% ($2to4$) of the production cost (Figure 2), losing sales cost from 25% to 50% ($7.5to15$) of the sale price (Figure 3), and two changes together (Figure 4).

Storage amount of all strategies still without change, thus the change only in the expected average cost. The expected average cost of ($S2$) is less than ($S1$). Thus, the second strategy ($S2$) with a slight losing sale becomes the best strategy, according to the expected average profit as shown in Figure 2.

Figure 3 shows $S1$ still is the best strategy, according to the expected average profit. That is due to the expected average cost of $S1$ still without change, strategy without losing sales. Meanwhile, there is a slight increase in the expected average cost of other strategies. The third strategy ($S3$) has a higher losing sales product than other strategies. Thus, increasing in the expected average cost of it is higher than other strategies.

The third strategy ($S3$) still the worst strategy, according to the expected average profit, despite it has the lowest expected average cost than other strategies as shown in Figure 4.

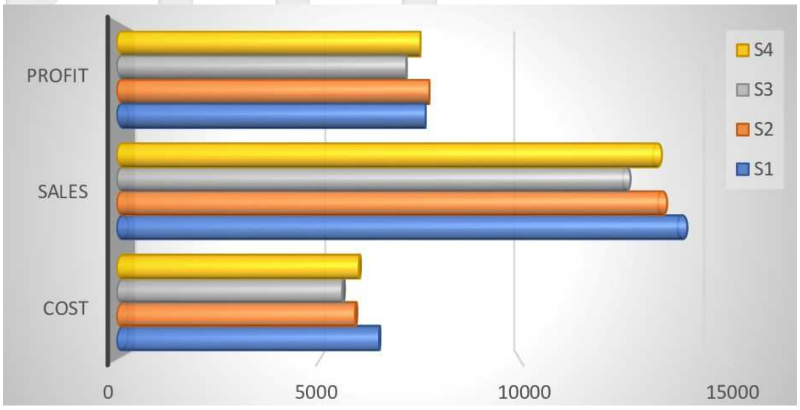

Figure 2: Expected averages of cost, sales and profit by changing of the holding cost.

Figure 3: Expected averages of cost, sales and profit by changing of lost sales cost.

Overall, the changing in the holding cost leads to slight losing sales is better than without losing sales. Numerical application focused on the suggested costs as a percent of production cost and sales price, thus flexible in applying model on the different industry fields.

5.2. Case 2: The pricing strategy

In this case, the demand is price dependent and in the same time is a random variable as shown in Table 13:

From Table 13, the demand depends on the sell price, so the demand divided into three categories. The categories represent sell price 300%, 250%, and 200% of the unit cost, respectively. Practically, the high sell price leads to decrease sells, and vice versa.

We take into consideration the first strategy only with the same production rates. Thus, the transition matrix of inventory level of the third category is as follows:

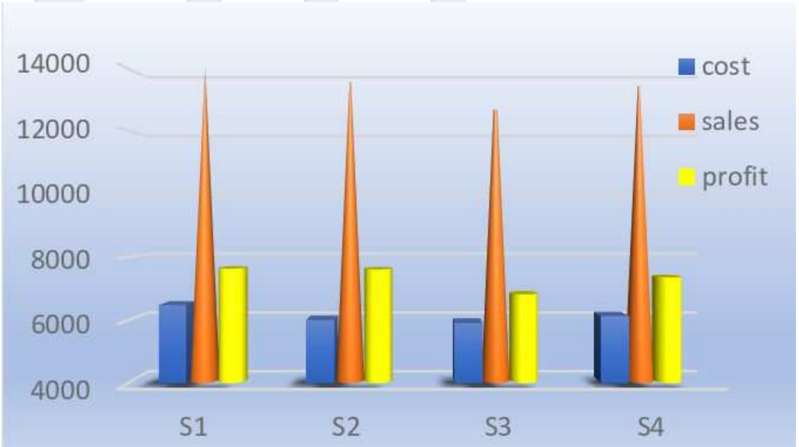

Figure 4: Expected averages of cost, sales and profit by changing of holding and lost sales costs.

Table 13: The probability distribution of demand (Case 2).

Demand	$P_r(D)$	$\sum P_r(D)$	a	Categories
50∼58	0.111	1	42	1
59∼72	0.026	0.358	35	2
73∼85	0.049	0.642	35	
86∼94	0.072	0.652	28	3
95∼100	0.058	0.348	28	

Table 14: The transition matrix of inventory level (case 2 (3)).

State	0	1	2
0	100	95∼99	86∼94
1	100	95∼99	86∼94
2	100	95∼99	86∼94

Table 15: The transition probability matrix (case 2 (3)).

State	0	1	2
0	0.058	0.29	0.652
1	0.058	0.29	0.652
2	0.058	0.29	0.652

The expected average cost (monthly cost) $\sum_{j=0}^{m} \mu_j \gamma_j =$

$$\sum_{j=0}^{m} \mu_j \gamma_j = 0.058 * 1192.8 + 0.29 * 1168.7 + 0.652 * 1121.7 = 1135.5\$.$$

The expected average sale (monthly) is as follows:

$$\sum_{j=0}^{m} \psi_j \gamma_j = 0.058 * 2800 + 0.29 * 2716 + 0.652 * 2520 = 2593.1\$.$$

The expected average profit is as follows:

$$O = 2593.1 - 1135.5 = 1453.6\$.$$

The transition matrix of inventory level of the second categories is as follows:

Table 16: The transition matrix of inventory level (case 2 (2)).

State	1	2
1	73∼85	59∼72
2	73∼85	59∼72

Table 17: The transition probability matrix (case 2 (2)).

State	1	2
1	0.642	0.358
2	0.642	0.358

The expected average profit (monthly) is as follows:

$$O = \sum_{j=0}^{m} \psi_j \gamma_j - \sum_{j=0}^{m} \mu_j \gamma_j = \quad (0.642 * 2765 + 0.358 * 2310) - (0.642 * 1043 + 0.358 * 948.5)$$
$$= 1592.9\$$$

The expected average profit (monthly) of the first categories is as follows:

$$O = \sum_{j=0}^{m} \psi_j \gamma_j - \sum_{j=0}^{m} \mu_j \gamma_j = 2268 - 871.1 = 1396.9\$.$$

Figure 5 shows the expected averages of cost, sales and profit of three categories.

From Figure 5, the best sell price is 250% of the unit cost ($C2$). This means the monthly demand is between 59 and 85 units. Meanwhile, the third category (C3) represents the highest cost due to the high amount of production compared with other

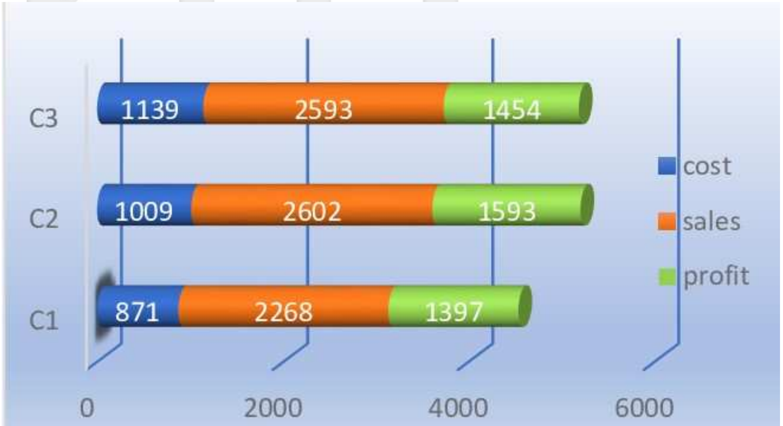

Figure 5: Expected averages of cost, sales and profit of the case 2.

categories. Choosing the pricing strategy depends also on the desire of the decision maker. More sales with a low profit or a high profit on account of the amount of sales.

6. Conclusion

In this paper, we studied a production-inventory system with defective and deteriorating items. Periodic review policy and Two cases of demand, were discussed; random variable and pricing dependent. We formulated a Markov decision model to describe the transition probabilities of the inventory level. Four production strategies were suggested; with losing sales, without losing sales, and mixed strategy. Defective and deteriorating products were disposed direct without rework. Also, check process was conducted on all product units. The expected average cost of the system included production, checking, holding, disposal and lost sales. The expected average profit and pricing strategy were determined.

Numerical application with suggested costs as a percent of production cost and sales price was conducted. Thus, flexible in applying model on different industry fields. The numerical results show arrange of strategies, according to the expected average profit. The arrangement is $S1$ (without lost sales), $S2$ (with slight lost sales), $S4$ (mixed), and the last strategy $S3$ (with significant lost sales). Choosing the production strategy not only depend of profit, but on production factors available. The effect of changing the holding cost, losing sales, as well as holding and losing sales together was illustrated. Sensitivity analysis shows the arrangement of strategies was affect by changing the holding cost. Meanwhile, the change in the losing sales cost led to decrease the expected average profit. The pricing strategy with random demand clarified the moderate price is better than high and low price. This work can extend to discuss the stochastic defective, reworking defective and deterioration products, and limited production and storage.

References

[1] Ahiska, S. S., and Kurtul, E. (2014). *Computers & Industrial Engineering Modeling and analysis of a product substitution strategy for a stochastic manufacturing / remanufacturing system*, Computers & Industrial Engineering, Vol.72, 1-11. https://doi.org/10.1016/j.cie.2014.02.015

[2] Assid, M., Gharbi, A., and Hajji, A. (2019). *Production and setup control policy for unreliable hybrid manufacturing- remanufacturing systems*, Journal of Manufacturing Systems, Vol.50, 103-118. https://doi.org/10.1016/j.jmsy.2018.12.004

[3] Bhowmick, J., and Samanta, G. P. (2011). A Deterministic Inventory Model of Deteriorating Items with Two Rates of Production, Shortages, and Variable Production Cycle. ISRN Applied Mathematics, 2011, 1-16. https://doi.org/10.5402/2011/657464

[4] Bouslah, B., Gharbi, A. and Pellerin, R. (2018). *Joint production , quality and maintenance control of a two-machine line subject to operation-dependent and quality-dependent failures*, International Journal of Production Economics, (October), 1-49. https://doi.org/10.1016/j.ijpe.2017.10.016

[5] Datta, T. K. (2016). *Inventory system with defective products and investment opportunity for reducing defective proportion*, Operational Research. https://doi.org/10.1007/s12351-016-0227-z

[6] Dhaiban, A. K. (2017). *A comparative study of stochastic quadratic programming and optimal control model in production-inventory system with stochastic demand*, Pesquisa Operacional, Vol.37, No.1, 193-208. https://doi.org/10.1590/0101-7438.2017.037.01.0193

[7] Dhaiban, A. K. and Aziz, N. (2019). *Stochastic demand of production-inventory system with shortage*, AIP Conference Proceedings, 2138 (1), 020003.

[8] Fang, C., Lai, M. and Huang, Y. (2017). *Production planning of new and remanufacturing products in hybrid production systems*, Computers & Industrial Engineering, Vol.108, 88-99. https://doi.org/10.1016/j.cie.2017.04.015

[9] Farsijani, H., Nikabadi, M. S. and Ayough, A. (2012). *A Simulated Annealing Approach to Optimize Multi-Products EPQ Model with Discrete Delivery Orders*, Imperfect Production Processes and Service Level Constraint, World Applied Sciences Journal, Vol.16, No.8, 1142-1157.

[10] Feng, H., Wu, Q., Muthuraman, K. and Deshpande, V. (2015). *Replenishment policies for multi-product stochastic inventory systems with correlated demand and joint-replenishment costs*, Production and Operations Management, Vol.24, No.4, 647-664. https://doi.org/10.1111/poms.12

[11] Hillier, F. S. and Lieberman, G. J. (2001). Introduction To Operations Research (seven). McGraw-Hill.

[12] Hu, M. and Yang, Y. (2014). *Modified Echelon (r, Q) policies with guaranteed performance bounds for stochastic serial inventory systems*, Operations Research, Vol.62, No.4, 812-828. https://doi.org/10.2139/ssrn.2085156

[13] Jin, X., Ni, J. and Koren, Y. (2011). *CIRP Annals - Manufacturing Technology Optimal control of reassembly with variable quality returns in a product remanufacturing system*, CIRP Annals - Manufacturing Technology, Vol.60, 25-28. https://doi.org/10.1016/j.cirp.2011.03.133

[14] Kim, J. S. and Jeong, W. C. (2012). *A coordination model under an order-up-to policy*, Informatica, Vol.23, No.2, 225-246. Retrieved from http://www.mii.lt/informatica/pdf/INFO860.pdf

[15] Kutzner, S. C. and Kiesmller, G. P. (2013). *Optimal control of an inventory-production system with state-dependent random yield*, European Journal of Operational Research, Vol.227, No.3, 444-452. https://doi.org/10.1016/j.ejor.2013.01.006

[16] Lee, S. and Kim, D. (2014). *An optimal policy for a single-vendor single-buyer integrated production - distribution model with both deteriorating and defective items*, Intern. Journal of Production Economics, Vol.147, 161-170. https://doi.org/10.1016/j.ijpe.2013.09.011

[17] Li, X., Li, Y. and Cai, X. (2015). *Remanufacturing and pricing decisions with random yield and random demand*, Computers and Operation Research, Vol.54, 195-203. https://doi.org/10.1016/j.cor.2014.01.005

[18] Liu, X., Banerjee, A. and Kim, S. L. (2009). Models For Retail Pricing And Customer Return Incentive For Remanufacturing A Product, POMS 20th Annual Conference, 1-30. Florida, USA.

[19] Moussawi-haidar, L., Salameh, M. and Nasr, W. (2014). *Effect of deterioration on the instantaneous replenishment model with imperfect quality items*, Applied Mathematical Modelling, Vol.38, No.24, 5956-5966. https://doi.org/10.1016/j.apm.2014.05.003

[20] Mubiru, K. (2014). *Optimal Ordering Policies for Inventory Problems in Supermarkets under Stochastic Demand: A Case Study of Milk Powder Product*, Proceedings of the 2014 International Conference on Industrial Engineering and Operations Management, 245-253. Bali, Indonesia.

[21] Nobil, A. H. and Sedigh, A. H. A. (2017). *An economic production quantity inventory model with a defective production system and uncertain uptime*, Int. J. Inventory Researc, Vol.4, No.(2/3), 132-147.

[22] Patel, H. (2019). *Pricing model for instantaneous deteriorating items with partial backlogging and different demand rates*, Uncertain Supply Chain Management, Vol.7, 97-108. https://doi.org/10.5267/j.uscm.2018.4.002

[23] Pervin, M., Roy, S. K. and Weber, G. W. (2018). *Analysis of inventory control model with shortage under time-dependent demand and time-varying holding cost including stochastic deterioration*, Annals of Operations Research, Vol.260, No.(1-2), 437-460. https://doi.org/10.1007/s10479-016-2355-5

[24] Priyan, S. and Uthayakumar, R. (2015). *An integrated production - distribution inventory system involving probabilistic defective and errors in quality inspection under variable setup cost*, International Transactions in Operational Research, 1-38. https://doi.org/10.1111/itor.12202

[25] Sarkar, B., Crdenas-barrón, L. E., Sarkar, M. and Laksono, M. (2014). *An economic production quantity model with random defective rate , rework process and backorders for a single stage production system*, Journal of Manufacturing Systems, Vol.33, No.3, 423-435. https://doi.org/10.1016/j.jmsy.2014.02.001

[26] Sharma, S., Sanjay, S. and Singh, S. R. (2018). *An inventory model for deteriorating items with expiry date and time varying holding cost*, International Journal of Procurement Management, Vol.11, No.5, 650-666.

[27] Soni, H. N. and Patel, K. A. (2012). *Optimal strategy for an integrated inventory system involving variable production and defective items under retailer partial trade credit policy*, Decision Support Systems, Vol.54, No.1, 235-247. https://doi.org/10.1016/j.dss.2012.05.009

[28] Tai, A. H. (2013). *Economic production quantity models for imperfect product and service with rework*, Computers & Industrial Engineering, Vol.66, No.4, 879-888.

[29] Tayyab, M., Sarkar, B. and Yahya, B. (2019). *Imperfect Multi-Stage Lean Manufacturing System with Rework under Fuzzy Demand*, Mathematics, Vol.7, 1-18. https://doi.org/10.3390/math7010013

[30] Uthayakumar, S. and Tharani, R. (2017). *An economic production model for deteriorating items and time dependent demand with rework and multiple production setups*, Journal of Industrial Engineering International, Vol.13, No.4, 499-512. https://doi.org/10.1007/s40092-017-0202-1

[31] Vercraene, S. and Gayon, J. (2011). *A two-stage production / inventory system with admission control of product returns*, International Conference on Industrial Engineering and Systems Management. METZ - FRANCE.

[32] Wen, H., Liu, M., Liu, C. and Liu, C. (2015). *Remanufacturing production planning with compensation function approximation method*, Applied Mathematics and Computation, Vol.256, 742-753. https://doi.org/10.1016/j.amc.2015.01.070

[33] Yin, K. K., Liu, H. and Johnson, N. E. (2002). *Markovian inventory policy with application to the paper industry*, Computers and Chemical Engineering, Vol.26, 1399-1413.

[34] Yu, J. C. P. (2013). *A collaborative strategy for deteriorating inventory system with imperfect items and supplier credits*, Intern. Journal of Production Economics, Vol.143, No.2, 403-409. https://doi.org/10.1016/j.ijpe.2011.11.018

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