International Journal of
Information
and
Management
Sciences

International Journal of Information and Management Sciences
31 (2020), 35-53. DOI:10.6186/IJIMS.202003_31(1).0003

Accelerated Life Testing for Double-Truncated General Half Normal Distribution

Hua Xin and Jian-ping Zhu

Northeast Petroleum University and Xiamen University

Abstract

The maximum likelihood estimation method and the Bayesian estimation method using Metropolis-Hastings Markov chain Monte Carlo (M-H MCMC) approach are investigated for estimating the parameters in the accelerated life test (ALT) model when the quality characteristic of product follows a double-truncated generalized half normal (DTGHN) distribution. To overcome the complexity by applying Fisher information matrix with the maximum likelihood estimates (MLEs) to obtaining the confidence intervals (CIs) of distribution quantiles, a bootstrap percentile method is used to obtain the CIs of distribution quantiles. The estimation performance of the proposed methods is evaluated by means of Monte Carlo simulations. Simulation results show that the proposed M-H MCMC method with non-informative prior distributions outperforms the maximum likelihood estimation method to obtain reliable MLEs of the ALT model parameters for the DTGHN distribution. An example about the stress-rupture life of Kevlar 49/epoxy is used to demonstrate the applications of the proposed methods and investigate the coverage probability of the bootstrap percentile CI for the distribution median at the normal-use condition.

Keywords: Bootstrap percentile method, maximum likelihood estimation, mean squared error, Metropolis-Hastings Markov chain Monte Carlo approach, Newton-Raphson method.

1. Introduction

When brittle martials, for example, the glasses, ceramics or some polymers, suffer sufficient stress in certain environments, the quality of materials gradually weak over time due to fracture, see Powell [16] and Wachtman [18]. Fatigue is a major problem for material weakness. Normally, the fatigue crack growth is highly related to the stress or cyclic load. Cooray and Ananda [4] are the pioneers to propose a generalized half normal (GHN) distribution to characterize the static fatigue crack growth under constant stress testing. They derived the static fatigue model and presented its properties, functional characteristics, maximum likelihood estimation process. Moreover, they studied the coverage probabilities of the maximum likelihood parameter estimation.

Pewsey [14] proposed inferential techniques for the GHN distribution, which was obtained via a line transformation from the GHN. Pewsey [15] developed bias correction method to construct bias-corrected CIs for the GHN distribution parameters. Cordeiro et al. [5] proposed the Kumaraswamy GHN distribution for modeling skewed positive data. The Kumaraswamy GHN distribution includes the half normal distribution and the GHN distribution, which was proposed by Cooray and Ananda [4], as special cases. They presented the explicit formulas of the probability density function (PDF), moments, generating and quantile functions, mean deviations and the moments of the order statistics. They also investigated the maximum likelihood estimation for the parameters and derive the expected information matrix. Olmos et al. [12] extended the GHN distribution proposed by Cooray and Ananda [4] and proposed a new GHN distribution by considering the quotient of two random variables, in which the one in the numerator follows a GHN distribution and the one in the denominator follows a power of the uniform distribution on (0,1), respectively. They provided the explicit expressions of the PDF of the new GHN distribution, and studied some properties and the moments of the new GHN distribution. By the way, they discussed the stochastic representation for the new model, and studied the maximum likelihood estimation and moment estimation methods for the new GHN distribution. Ahmadi et al. [1] studied the estimation method for obtaining the MLEs of the parameters in the GHN distribution based on progressive type II censoring samples. They also proposed methods to obtain Bayes estimates through using different symmetric and asymmetric loss functions such as squared error, LINEX or general entropy.

Nogales and Pérez [11] proposed unbiased estimators for the GHN distribution parameters, and they have shown that their unbiased estimators outperform some existing maximum likelihood estimators. Cordeiro et al. [6] proposed a new GHN distribution, named odd log-logistic GHN distribution for describing fatigue lifetime data. They also studied the maximum likelihood estimation method for the odd log-logistic GHN distribution. Wang [21] studied the maximum likelihood estimation and developed bias correction methods for obtaining reliable estimates of the GHN distribution parameters. He also studied the methods to obtain the unweighted and weighted least squares estimates for the GNH distribution parameters. Altun et al. [2] introduced a new extension of the GNH distribution and they assessed the performance of the maximum likelihood estimators of the model parameter through using simulations. Pescim et al. [13] proposed the beta GNH distribution, which is another generalized version of the half normal distribution.

He et al. [7] proposed a data-driven method for capacity fast prediction and estimating the residual useful life estimation for high-capacity valve regulated lead acid batteries. In their study, the ALT method is used to save test time. Wang et al. [20] studied simple and efficient methods to estimate the coefficients of the ALT model. They assumed that both scale and shape parameters of the Weibull distribution vary along with the stress levels of stress variable. Then they proposed maximum likelihood estimation methods to obtain the MLEs of model parameters. Moreover, they studied methods to reduce the estimation bias when estimating the shape parameter. The ALT method have been widely studied in the literature. Lawless [8] and Nelson [10] provided comprehensive discussions for using the censoring and ALT methods in life testing applications. Wang and Kececioglu [19] used Weibull log-linear model to fit ALT data and they proposed inference methods for parameter estimation. Srivastava and Savita [17] proposed an ALT plan for two-component parallel systems when the ramp-stress loading is used for masked data. Meeker and Escobar [9] provided comprehensive discussions and summary about using the censoring, ALT and accelerated degradation test methods in recent life testing applications. A list of the full names of technique terms, which are used in this study, is given below:

The rest of this paper is organized as follows: In Section 2, the motivation is given to deliver the three goals of this study. The DTGHN distribution is addressed in Section 3 and the statistical model of ALT for the DTGHN distribution is obtained. The maximum likelihood estimation procedure is derived. Moreover, the Bayes estimators via using M-H MCMC method with non-informative prior distributions for the generalized ALT model parameters are analytically obtained. The bootstrap percentile method is used to find the CI of the quality parameter based on ALT samples. An algorithm is given to implement the proposed bootstrap percentile method. In Section 4, a numerical study is conducted with the Monte Carlo simulations to evaluate the estimation performance of the maximum likelihood estimators and Bayes estimators. Then a reliable estimation procedure is recommended. The sensitivity of the proposed M-H MCMC procedure for model misspecification is also studied. An example regarding the stress-rupture life of Kevlar 49/epoxy is used to demonstrate the applications of the proposed methods in Section 5, and the bootstrap percentile CI of the median lifetime of the DTGHN distribution at the normal-use condition is found and the coverage probability is investigated. Some conclusions are given in Section 6.

Technique term	**Full name**
ALT	accelerated life test
CI	confidence interval
DTGHN	double-truncated generalized half normal
GHN	generalized half normal
M-H MCMC	Metropolis-Hastings Markov chain Monte Carlo
MLE	maximum likelihood estimate
PDF	probability density function
QN	quasi-Newton algorithm
SN	skew-normal
GSN	generalized skew-normal
TGSN	truncated generalized skew-normal

2. Motivation

The ALT method contains benefit for saving testing time, and the GHN distribution is a good distribution for describing the fatigue crack growth of materials. In this study, three goals are investigated for the inference methods on the ALT model parameters when the fatigue crack growth of materials follows a GHN distribution:

(1) The GHN is a good candidate distribution to characterize the fatigue crack growth of materials. It is important to establish a generalized ALT inference process for a generalized distribution class, which contains the GHN distribution as special case. Because of the highly reliable property of today's devices, we need a more complicate ALT model to characterize the relationships between the distribution parameters and the stress. In this study, two underlying distribution parameters are allowed to link with the stress levels. The proposed ALT model contains the widely used ALT model, which only links the scale parameter to the stress (see Nelson [10] and Meeker and Escobar [9]), as special case. The DTGHN distribution is a generalized family that includes the lower-truncated GHN, upper-truncated GHN and GHN distributions as special cases. Hence, the DTGHN distribution is used as the underlying distribution to characterize the fatigue crack growth of materials to establish the inference process for the generalized ALT model. The proposed methods not only can be applied to implement reliability assessments for the fatigue crack growth of materials, the proposed methods can also be applied to other related engineering applications, in which the DTGHN distribution is used to be the underlying distribution to characterize the quality characteristic of products, for saving test time and cost.

(2) To obtain reliable estimates for the generalized ALT model that mentioned in Goal 1, a maximum likelihood estimation procedure is analytically presented. We use quasi-Newton algorithm to implement the Newton-Raphson method to obtain the MLEs of the generalized ALT model parameters. In order to overcoming the divergence problem during searching the MLEs in this study, a Bayesian estimation procedure is also developed to obtain the Bayes estimators of the model parameters through using non-informative prior distributions. The obtained Bayes estimates are close to the MLEs of the model parameters. Because of no explicit expressions for the Bayes estimators of the parameters, the Bayes estimates are searched via using an M-H MCMC method. The purpose of the second goal is to find a simple and stable parameter estimation procedure to obtain reliable estimates of the ALT model parameters for the DTGHN distribution.

(3) We would like to provide a simple interval inference method for estimating the quality parameter based on ALT samples. Sampling error has impact on the quality of point estimates. Hence a CI estimation method is needed to investigate the impact of the sampling error on the point estimate. We proposed a bootstrap procedure to obtain the CIs of the ALT model parameters. The proposed bootstrap procedure is easy to implement.

3. Statistical Model and Estimation Method

In this section, we investigate the DTGHN distribution and the ALT model for the DTGHN distribution. Moreover, the maximum likelihood estimation procedure and the Bayesian estimation method using M-H MCMC method to infer the ALT model parameters are analytically derived. The bootstrap percentile CI of the quality parameter is presented with an algorithm.

3.1. Statistical model

Let the quality characteristic of a product T follow a GHN distribution, which has the PDF

$$f(t;\alpha,\eta)=\sqrt{\frac{2}{\pi}}\Big(\frac{\alpha}{t}\Big)\Big(\frac{t}{\eta}\Big)^{\alpha}\exp\Big\{-\frac{1}{2}\Big(\frac{t}{\eta}\Big)^{2\alpha}\Big\},\quad \alpha,\eta,t>0, \tag{3.1}$$

where α is shape parameter and η is scale parameter. Let $\theta=\eta^{-2\alpha}$, then the PDF in Equation (3.1) can be represented by

$$f(t;\alpha,\theta)=\sqrt{\frac{2}{\pi}}\alpha\sqrt{\theta}t^{\alpha-1}\exp\Big\{-\frac{1}{2}\theta t^{2\alpha}\Big\},\quad \alpha,\theta,t>0. \tag{3.2}$$

The cumulative distribution function (CDF) based on the PDF in Equation (3.2) can be presented by

$$F(t;\alpha,\theta)=2\Phi[\sqrt{\theta}t^{\alpha}]-1=1-2\Phi[-\sqrt{\theta}t^{\alpha}],\quad \alpha,\theta,t>0 \tag{3.3}$$

where $\Phi(\cdot)$ is the CDF of standard normal distribution. The survival function can be presented by

$$S(t;\alpha,\theta)=1-F(t;\alpha,\theta)=2\Phi[-\sqrt{\theta}t^{\alpha}],\quad \alpha,\theta,t>0. \tag{3.4}$$

We can extend the $F(t;\alpha,\theta)$ of GHN distribution defined in Equations (3.3) and (3.4) to a DTGHN distribution by

$$\begin{aligned}G(x)\equiv G(x;\alpha,\theta)=&\frac{F(x;\alpha,\theta)-F(\mu;\alpha,\theta)}{F(v;\alpha,\theta)-F(\mu;\alpha,\theta)}\\=&\frac{\Phi[-\sqrt{\theta}\mu^{\alpha}]-\Phi[-\sqrt{\theta}x^{\alpha}]}{d(\mu,v;\alpha,\theta)},\quad 0<\mu\le x\le v,\end{aligned} \tag{3.5}$$

where μ and v are the lower and upper truncated bounds, respectively and $d(\mu,v;\alpha,\theta)=\Phi[-\sqrt{\theta}\mu^{\alpha}]-\Phi[-\sqrt{\theta}v^{\alpha}]$. Denote the DTGHN distribution in Equation (3.5) by DTGHN(α,θ). The DTGHN(α,θ) is a generalization of the GHN distribution. When $\mu=0$ and $v\to\infty$, $G(x)=F(t;\alpha,\theta)$; when $\mu\to 0$, $G(x)$ is the upper truncated GHN distribution and when $v\to\infty$, $G(x)$ is the lower truncated GHN distribution. The PDF of $G(x)$ can be obtained by

$$g(x;\alpha,\theta)=\frac{\sqrt{\theta}\alpha x^{\alpha-1}\Phi[-\sqrt{\theta}x^{\alpha}]}{d(\mu,v;\alpha,\theta)},\quad 0<\mu\le x\le v, \tag{3.6}$$

where $\Phi(\cdot)$ is the PDF of the standard normal distribution. The values of μ and v can be determined based on the knowledge from historical data. If practitioners have no idea or lack historical information to determine the values of μ and v, they can take $\mu \to 0$ and $v \to \infty$ to use the DTGHN distribution for practical applications.

3.2 The accelerated life test model and estimation method

Assume that the lifetime products are tested using an ALT with stress s, which has the levels $s_1 \leq s_2 \leq \cdots \leq s_m$. Denote the relationship of the parameters in the DTGHN distribution with the stress levels by $\theta_i \equiv \theta(s_i) = b_0 + b_1 s_i$ and $\alpha_i \equiv \alpha(s_i) = a_0 + a_1 s_i$. Total n_i lifetime units are used for life testing under the stress s_i for $i = 1, 2, \ldots, m$. Let $\Theta = (a_0, a_1, b_0, b_1)$. Then the likelihood function based on the ALT sample $x = \{x_{ij}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n_i\}$ can be obtained by

$$\begin{aligned} L(\Theta; \boldsymbol{x}) =& \prod_{i=1}^{m} \prod_{j=1}^{n_i} g(x_{ij}; \Theta) \\ =& \prod_{i=1}^{m} \prod_{j=1}^{n_i} \frac{\sqrt{\theta_i} \alpha_i x_{ij}^{\alpha_i - 1} \Phi[-\sqrt{\theta_i} x_{ij}^{\alpha_i}]}{d(\mu, v; \alpha_i, \theta_i)}. \end{aligned} \tag{3.7}$$

Moreover, the log-likelihood function based on the Equation (3.7) can be obtained by

$$\begin{aligned} \ell(\Theta; \boldsymbol{x}) =& \sum_{i=1}^{m} n_i \Big(\frac{1}{2} \ln \theta_i + \ln \alpha_i \Big) + \sum_{i=1}^{m} (\alpha_i - 1) \sum_{j=1}^{n_i} \ln x_{ij} \\ &+ \sum_{i=1}^{m} \sum_{j=1}^{n_i} \ln \Phi[-\sqrt{\theta_i} x_{ij}^{\alpha_i}] - \sum_{i=1}^{m} n_i \ln\{d(\mu, v; \alpha_i, \theta_i)\}. \end{aligned} \tag{3.8}$$

Denote the MLEs of the parameters a_0, a_1, b_0 and b_1 by $\hat{a}_0$, $\hat{a}_1$, $\hat{b}_0$ and $\hat{b}_1$, respectively. Let $\hat{\theta}_i = \hat{b}_0 + \hat{b}_1 s_i$, $\hat{\alpha}_i = \hat{a}_0 + \hat{a}_1 s_i$ and $\hat{\Theta} = \hat{a}_0, \hat{a}_1, \hat{b}_0, \hat{b}_1$. Then the MLEs $\hat{a}_0, \hat{a}_1, \hat{b}_0$ and $\hat{b}_1$ can be obtained by simultaneously solving the log-likelihood equations $\ell_{a_0} = \frac{\partial \ell(\Theta; x)}{\partial a_0}\Big|_{\Theta = \hat{\Theta}} = 0$, $\ell_{a_1} = \frac{\partial \ell(\Theta; x)}{\partial a_1}\Big|_{\Theta = \hat{\Theta}} = 0$, $\ell_{b_0} = \frac{\partial \ell(\Theta; x)}{\partial b_0}\Big|_{\Theta = \hat{\Theta}} = 0$ and $\ell_{b_1} = \frac{\partial \ell(\Theta; x)}{\partial b_1}\Big|_{\Theta = \hat{\Theta}} = 0$, where

$$\begin{aligned} \ell_{a_0} =& \sum_{i=1}^{m} \frac{n_i}{\hat{\alpha}_i} + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \ln x_{ij} - \sum_{i=1}^{m} \hat{\theta}_i \sum_{j=1}^{n_i} x_{ij}^{2\hat{\alpha}_i} \ln x_{ij} \\ &+ \sum_{i=1}^{m} n_i \frac{\Phi(-\sqrt{\hat{\theta}_i} \mu^{\hat{\alpha}_i}) \sqrt{\hat{\theta}_i} \mu^{\hat{\alpha}_i} \ln \mu - \Phi(-\sqrt{\hat{\theta}_i} v^{\hat{\alpha}_i}) \sqrt{\hat{\theta}_i} v^{\hat{\alpha}_i} \ln v}{d(\mu, v; \hat{\alpha}_i, \hat{\theta}_i)}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \ell_{a_1} =& \sum_{i=1}^{m} \frac{n_i s_i}{\hat{\alpha}_i} + \sum_{i=1}^{m} s_i \sum_{j=1}^{n_i} \ln x_{ij} - \sum_{i=1}^{m} s_i \hat{\theta}_i \sum_{j=1}^{n_i} x_{ij}^{2\hat{\alpha}_i} \ln x_{ij} \\ &+ \sum_{i=1}^{m} n_i s_i \frac{\Phi(-\sqrt{\hat{\theta}_i} \mu^{\hat{\alpha}_i}) \sqrt{\hat{\theta}_i} \mu^{\hat{\alpha}_i} \ln \mu - \Phi(-\sqrt{\hat{\theta}_i} v^{\hat{\alpha}_i}) \sqrt{\hat{\theta}_i} v^{\hat{\alpha}_i} \ln v}{d(\mu, v; \hat{\alpha}_i, \hat{\theta}_i)}, \end{aligned} \tag{3.10}$$

$$\ell_{b_0} = \frac{1}{2}\sum_{i=1}^{m}\frac{n_i}{\hat{\alpha}_i} - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i} x_{ij}^{2\hat{\alpha}_i} + \frac{1}{2}\sum_{i=1}^{m}\frac{n_i}{\sqrt{\hat{\theta}_i}}\frac{\Phi(-\sqrt{\hat{\theta}_i}\mu^{\hat{\alpha}_i})\mu^{\hat{\alpha}_i} - \Phi(-\sqrt{\hat{\theta}_i}v^{\hat{\alpha}_i})v^{\hat{\alpha}_i}}{d(\mu, v; \hat{\alpha}_i, \hat{\theta}_i)}, \tag{3.11}$$

$$\ell_{b_1} = \frac{1}{2}\left\{\sum_{i=1}^{m}\frac{n_i s_i}{\hat{\theta}_i} - \sum_{i=1}^{m} s_i \sum_{j=1}^{n_i} x_{ij}^{2\hat{\alpha}_i} + \sum_{i=1}^{m}\frac{n_i s_i}{\sqrt{\hat{\theta}_i}}\frac{\Phi(-\sqrt{\hat{\theta}_i}\mu^{\hat{\alpha}_i})\mu^{\hat{\alpha}_i} - \Phi(-\sqrt{\hat{\theta}_i}v^{\hat{\alpha}_i})v^{\hat{\alpha}_i}}{d(\mu, v; \hat{\alpha}_i, \hat{\theta}_i)}\right\}, \tag{3.12}$$

The Equations (3.9) to (3.12) are very complicated. It is not easy to search the MLEs $\hat{a}_0$, $\hat{a}_1$, $\hat{b}_0$ and $\hat{b}_1$ by directly solving the four log-likelihood equations. Normally, algorithms for implementing Newton-Rapson method can be used to obtain the MLEs. But the estimating results could be unstable due to it is very difficult to set up four proper initial parameter solutions to implement the Newton-Rapson method. In this study, we propose a MCMC approach using M-H algorithm to obtain Bayes estimates of the ALT model parameters, in which non-informative prior distributions of the model parameters are applied. The resulting Bayes estimates of the parameters are close to the MLEs. Considering the prior PDF for Θ :

$$\pi(\Theta) = \pi(a_0, a_1, b_0, b_1) = \pi(a_0) \times \pi(a_1) \times \pi(b_0) \times \pi(b_1), \tag{3.13}$$

where $\pi(a_0) \propto c_1$, $\pi(a_1) \propto c_2$, $\pi(b_0) \propto c_3$ and $\pi(b_1) \propto c_4$, c_i is constant for $i = 1, 2, 3,$ and 4. The posterior PDF can be obtained by

$$\pi(\Theta; \boldsymbol{x}) \propto L(\Theta; \boldsymbol{x}) \times \pi(\Theta). \tag{3.14}$$

Because $\pi(\Theta) \propto$ constant, then we obtain

$$\pi(\Theta; \boldsymbol{x}) \propto L(\Theta; \boldsymbol{x}). \tag{3.15}$$

We take into account the squared loss function to implement the proposed Bayesian estimation procedure. Hence, the Bayes estimator of parameter can be the mean of the posterior distribution. Because no explicit expressions for the Bayes estimators of the ALT model parameters, the M-H MCMC approach is applied to searching the Bayes estimates. It is noted that the major contribution in the posterior distribution in Equation (3.15) comes from the likelihood function, and hence the Bayes estimate of Θ is close to the MLE of Θ in this study. The M-H MCMC procedure can be implemented with the following algorithm:

Algorithm 1: M-H MCMC procedure
Initial Step: Establish the initial states, $a_0^{(0)}$, $a_1^{(0)}$, $b_0^{(0)}$ and $b_1^{(0)}$, for parameters a_0, a_1, b_0 and b_1.

Step 1: Propose the transition probabilities (or named proposals) $q_1(a_0^*; a_0^{(i)})$ according to $a_0^{(i)}$ to a_0^*, propose the transition probabilities $q_2(a_1^*; a_1^{(i)})$ according to $a_1^{(i)}$ to a_1^*, propose the transition probabilities $q_3(b_0^*; b_0^{(i)})$ according to $b_0^{(i)}$ to b_0^* and $q_4(b_1^*; b_1^{(i)})$ according to $b_1^{(i)}$ to b_1^*.

Step 2: Implement Step 2.1−Step 2.4 N times for $i = 0, 1, 2, \ldots, N$, where N is a huge number.

Step 2.1: Generate a_0^* from $q_1(a_0^*; a_0^{(i)})$, and generate u from $\mathrm{U}(0,1)$, where $\mathrm{U}(0,1)$ denotes the uniform distribution over the interval $(0,1)$. Update $a_0^{(i+1)}$ according to the following condition:

$$a_0^{(i+1)} = \begin{cases} a_0^*, & \text{if } u \le \min\left\{1, \dfrac{\pi(a_0^*; a_1^{(i)}, b_0^{(i)}, b_1^{(i)}, \boldsymbol{x})}{\pi(a_0^{(i)}; a_1^{(i)}, b_0^{(i)}, b_1^{(i)}, \boldsymbol{x})} \dfrac{q_1(a_0^{(i)}; a_0^*)}{q_1(a_0^*; a_0^{(i)})}\right\}, \\ a_0^{(i)}, & \text{otherwise.} \end{cases} \tag{3.16}$$

Step 2.2: Generate a_1^* from $q_2(a_1^*; a_1^{(i)})$, and generate u from $\mathrm{U}(0,1)$. Update $a_1^{(i+1)}$ according to the following condition:

$$a_1^{(i+1)} = \begin{cases} a_1^*, & \text{if } u \le \min\left\{1, \dfrac{\pi(a_1^*; a_0^{(i+1)}, b_0^{(i)}, b_1^{(i)}, \boldsymbol{x})}{\pi(a_1^{(i)}; a_0^{(i+1)}, b_0^{(i)}, b_1^{(i)}, \boldsymbol{x})} \dfrac{q_2(a_1^{(i)}; a_1^*)}{q_2(a_1^*; a_1^{(i)})}\right\}, \\ a_1^{(i)}, & \text{otherwise.} \end{cases} \tag{3.17}$$

Step 2.3: Generate b_0^* from $q_3(b_0^*; b_0^{(i)})$, and generate u from $\mathrm{U}(0,1)$. Update $b_0^{(i+1)}$ according to the following condition:

$$b_0^{(i+1)} = \begin{cases} b_0^*, & \text{if } u \le \min\left\{1, \dfrac{\pi(b_0^*; a_0^{(i+1)}, a_0^{(i+1)}, b_1^{(i)}, \boldsymbol{x})}{\pi(b_0^{(i)}; a_0^{(i+1)}, a_1^{(i+1)}, b_1^{(i)}, \boldsymbol{x})} \dfrac{q_3(b_0^{(i)}; b_0^*)}{q_3(b_0^*; b_0^{(i)})}\right\}, \\ b_0^{(i)}, & \text{otherwise.} \end{cases} \tag{3.18}$$

Step 2.4: Generate b_1^* from $q_4(b_1^*; b_1^{(i)})$, and generate u from $\mathrm{U}(0,1)$. Update $b_1^{(i+1)}$ according to the following condition:

$$b_1^{(i+1)} = \begin{cases} b_1^*, & \text{if } u \le \min\left\{1, \dfrac{\pi(b_1^*; a_0^{(i+1)}, a_1^{(i+1)}, b_0^{(i+1)}, \boldsymbol{x})}{\pi(b_1^{(i)}; a_0^{(i+1)}, a_1^{(i+1)}, b_0^{(i+1)}, \boldsymbol{x})} \dfrac{q_4(b_1^{(i)}; b_1^*)}{q_4(b_1^*; b_1^{(i)})}\right\}, \\ b_1^{(i)}, & \text{otherwise.} \end{cases} \tag{3.19}$$

Step 3: The Bayes estimates can be obtained by $\hat{a}_{0B} = \dfrac{\sum_{i=M+1}^{N} a_0^{(i)}}{N-M}$, $\hat{a}_{1B} = \dfrac{\sum_{i=M+1}^{N} a_1^{(i)}}{N-M}$, $\hat{b}_{0B} = \dfrac{\sum_{i=M+1}^{N} b_0^{(i)}}{N-M}$ and $\hat{b}_{1B} = \dfrac{\sum_{i=M+1}^{N} b_1^{(i)}}{N-M}$ based on the squared loss functions $L(\hat{a}_{0B}, a_0) = (\hat{a}_{0B} - a_0)^2$, $L(\hat{a}_{1B}, a_0) = (\hat{a}_{1B} - a_1)^2$, $L(\hat{b}_{0B}, b_0) = (\hat{b}_{0B} - b_0)^2$ and $L(\hat{b}_{1B}, b_1) = (\hat{b}_{1B} - b_1)^2$, respectively, where the first M chains are used for burn-in and the burn-in chains are removed from the parameter estimation.

In practice, we can select symmetric transition probability functions to reduce computation loading. In this study, the uniform distribution is used for characterizing the transition probability functions $q_i(\cdot)$ for $i = 1, 2, 3$ and 4.

3.3. The bootstrap method for interval estimation

The CIs of the parameters a_0, a_1, b_0 and b_1 can be obtained by using the MLEs and observed Fisher information matrix, which contains the "second derivatives" of the log-likelihood function with respect to the parameters a_0, a_1, b_0 and b_1 as components. Because the first derivatives of the log-likelihood function in Equations (3.9) to (3.12) are very complicated, it is difficult to obtain the explicit form of Fisher information matrix. This fact makes that the observed Fisher information matrix conservative for practical use. Hence, we recommend to use bootstrap percentile method to obtain the CIs of the function of the parameters a_0, a_1, b_0 and b_1, denoted it by $\delta(\equiv \delta(\Theta))$. The δ can be the quality parameter, for example the distribution quantiles. Based on the invariant property of the MLEs, the MLE of δ is denoted by $\hat{\delta}(\equiv \delta(\hat{\Theta}))$ in which $\hat{\Theta}$ is the MLE of Θ.

Algorithm 2: The bootstrap procedure

Step 1: Obtain the MLEs of a_0, a_1, b_0 and b_1 based on the random sample, which has n observations taken from the DTGHN(α_i, θ_i) with the stress equations $\theta_i = (b_0 + b_1 s_i)$ and $\alpha_i = (a_0 + a_1 s_i)$ for $i = 1, 2, \ldots, m$, and denote the vector of MLEs by $\hat{\Theta} = (\hat{a}_0, \hat{a}_1, \hat{b}_0, \hat{b}_1)$.

Step 2: Generate bootstrap samples, each sample with n observations, from the DTGHN$(\hat{\theta}_i, \hat{\alpha}_i)$, where $\hat{\theta}_i = \hat{b}_0 + \hat{b}_1 s_i$ and $\hat{\alpha}_i = \hat{a}_0 + \hat{a}_1 s_i$ for $i = 1, 2, \ldots, m$. Obtain the MLEs of a_0, a_1, b_0 and b_1 based on the bootstrap samples and denote than by $\hat{\Theta}^* = (\hat{a}_0^*, \hat{a}_1^*, \hat{b}_0^*, \hat{b}_1^*)$.

Step 3: Repeat Step 2 B times, and denote the bootstrap estimates of Θ by $\hat{\Theta}^{*(j)}$ for $j = 1, 2, \ldots, B$. The bootstrap estimates of $\delta(\Theta)$ are denoted by $\hat{\delta}^{*(j)} \equiv \delta(\hat{\Theta}^{*(j)})$ for $j = 1, 2, \ldots, B$. Denote the bootstrap empirical sampling distribution of $\hat{\delta}$ by $\hat{G}_B$, which can be constructed using the bootstrap estimates $\hat{\delta}^{*(j)}$ for $j = 1, 2, \ldots, B$.

Step 4: The $(1-2\gamma) \times 100\%$ bootstrap percentile CI is $(\hat{\delta}_L^*, \hat{\delta}_U^*)$, where $\hat{\delta}_L^*$ and $\hat{\delta}_U^*$ is the 100γth percentile and $100(1-\gamma)$th percentile of $\hat{G}_B$.

In reliability evaluation studies, we often are interested in evaluating the $100p$th percentile of the lifetimes of the products. If the underlying distribution at the normal-use condition is DTGHN(α_0, θ_0), then we can have $\delta = x_p$. It can be shown that the $100p$th percentile x_p can be presented by

$$x_p = \left(-\frac{1}{\sqrt{\theta_0}} \Phi^{-1}\{\Phi(-\sqrt{\theta_0}\mu^{\alpha_0}) - p \times d(\mu, v; \alpha_0, \theta_0)\} \right)^{\frac{1}{\alpha_0}}. \tag{3.20}$$

In particular, when $\mu = 0$ and $v = \infty$, we obtain $d(\mu, v; \alpha_0, \theta_0) = 1/2$ and

$$x_p = \left(-\frac{1}{\sqrt{\theta_0}} \Phi^{-1}\Big\{\frac{1-p}{2}\Big\} \right)^{\frac{1}{\alpha_0}} = \left(\frac{1}{\sqrt{\theta_0}} \Phi^{-1}\Big\{\frac{1+p}{2}\Big\} \right)^{\frac{1}{\alpha_0}}. \tag{3.21}$$

Equation (3.21) is the 100pth percentile of the GHN distribution at the normal-use condition. The implementation of the proposed methods can be summarized in the following algorithm:

Algorithm 3: The implementation of the proposed method

Step 1: Obtain the normalized stress levels and denote them by $s_0 \leq s_1 \leq s_2 \leq \cdots \leq s_m$ in which the normal-use stress $s_0 = 0$ and the highest stress $s_m = 1$.

Step 2: Obtain the Bayes estimates of the model parameters via using the proposed M-H MCMC procedure in Algorithm 1. Then obtain the Bayes estimate of δ.

Step 3: Obtain the $(1 - 2\gamma) \times 100\%$ bootstrap percentile CI of δ by using the proposed bootstrap procedure in Algorithm 2.

4. Monte Carlo Simulations

A simulation study is conducted in this section to study the estimation performance of the proposed maximum likelihood estimation and M-H MCMC methods. Moreover, the estimation performance of the proposed M-H MCMC method is compared with the maximum likelihood estimation. When implement the maximum likelihood estimation, the MLEs of the ALT model parameters are obtained by using the Newton-Raphson method with the QN algorithm, which was proposed by Byrd et al. [3] is an efficient algorithm, which uses a limited-memory modification to implement the Newton-Raphson method. The QN method allows box constraint for searching the solution of parameter in an interval. The initial solutions of parameters must satisfy the box constraints. Denote the normal-use condition stress level and the highest stress level by $s_0 = 0$ and $s_H = 1$. The highest stress level must be free of the over stress condition and decided based on the knowledge of engineers. In this study, we consider two normalized stress

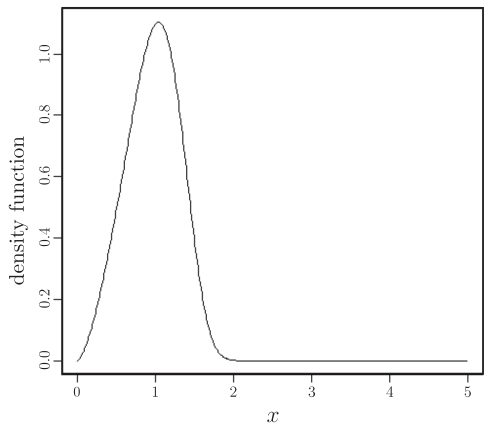

Figure 1: The density curve of the DTGHN($\alpha_0 = a_0, \theta_0 = b_0$) with $u = 0$, $v = 5$, $a_0 = 2.5$ and $b_0 = 0.5$ under the normal-use condition.

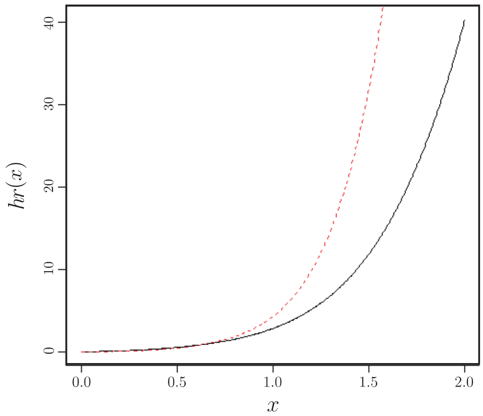

Figure 2: The hazard rates of the (a) DTGHN(α_1, θ_1) (solid line) and (b) DTGHN(α_2, θ_2) (dash line).

levels with $s = (s_1, s_2) = (s_L, s_H) = (0.3, 1)$ for the ALT. Assume that $n = 10, 20$, and 30 components are used for implementing the ALT. The quality characteristics of components follow a DTGHN$(\alpha_i = a_0 + a_1 s_i,\ \theta_i = b_0 + b_1 s_i)$, which was defined in the Equations (3.5) and (3.6) with $u = 0$, $v = 5$, $a_0 = 2.5$, $a_1 = 1$, $b_0 = 0.5$ and $b_1 = 0.25$ for $i = 1$ and 2. The density curve at the normal-use condition of stress is given in Figure 1, and the hazard rates under two used stress levels are plotted in Figure 2. When the stress is at the normal-use condition, we have $\alpha_0 = a_0$, and $\theta_0 = b_0$. From Figure 2 we can see that the hazard rate increases when the stress increases to s_2 from the lower stress s_1.

For doing a fair comparison for the M-H MCMC method with the Newton-Raphson method, we search the estimates of parameters a_0, a_1, b_0 and b_1 over the same domains for the parameters with $D_{a_0} = \{0.5 \leq a_0 \leq 5\}$, $D_{a_1} = \{0.5 \leq a_0 \leq 5\}$, $D_{b_0} = \{0.01 \leq b_0 \leq 1.5\}$ and $D_{b_1} = \{0.01 \leq b_1 \leq 1.5\}$, respectively. Because we cannot know the proper initial solutions of parameters in most real cases when applying Newton-Raphson method to searching the MLEs of the parameters, the initial solutions of a_0, a_1, b_0 and b_1 are randomly generated from the uniform distributions that are defined on the domains D_{a_0}, D_{a_1}, D_{b_0} and D_{b_1}, respectively. When applying the M-H MCMC method to searching the estimates of parameters, we use the uniform distributions over the domains D_{a_0}, D_{a_1}, D_{b_0} and D_{b_1} as the transition probabilities q_1, q_2, q_3 and q_4.

Consider $N = 8000$ chains to implement the M-H MCMC method. Moreover, we drop the first 20% chains for burn-in, that is, the first 1600 chains are drop out from the Markov chains for burn-in in each iteration run of simulations. Then the Bayes estimates of a_0, a_1, b_0 and b_1 are obtained by using the remainder 6400 Markov chains. Because non-informative prior distributions are used in the M-H MCMC method and we use uniform distributions as the transition probabilities, the resulting Bayes estimates of a_0, a_1, b_0 and b_1 are close to the MLEs.

Figures 3 to 6 show the box plots of 10000 obtained MLEs and Bayes estimates of the parameters a_0, a_1, b_0 and b_1 for different sample sizes. We can find that almost all

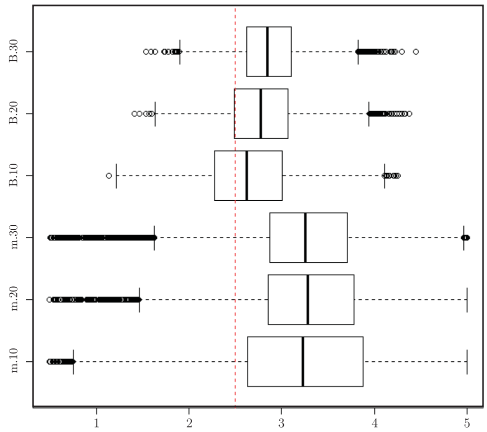

Figure 3: The box plots of 10000 estimates of a_0, in which the "m.n" and "B.n" denote the MLE and Bayes estimate of a_0, respectively, that are obtained based on samples each of size n.

box length decreases as the sample size increases. The only exception is the MLEs in the Figure 6. Because the Newton-Raphson method requires precisely initial solutions of parameters to obtain reliable MLE for complicated log-likelihood function, the MLE becomes unstable if improper initial solutions are used. In this simulation study, the initial solutions of parameters are uniformly generated from the domains D_{a_0}, D_{a_1}, D_{b_0} and D_{b_1}, respectively. We cannot guarantee that we can always obtain proper initial solutions of the parameters. This drawback is the major difficulty to implement Newton-Raphson method for searching the reliable MLEs of the ALT model parameters.

From Figures 3 to 6 we find that the M-H MCMC method outperforms the Newton-Raphson method to obtain reliable estimates of the model parameters. Compared with the MLEs, the Bayes estimates have less dispersion and almost all the medians of Bayes

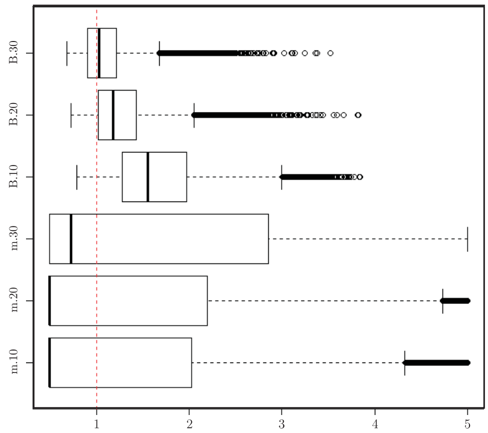

Figure 4: The box plots of 10000 estimates of a_1, in which the "m.n" and "B.n" denote the MLE and Bayes estimate of a_1, respectively, that are obtained based on samples each of size n.

estimates of the ALT model parameters are closer to their true values than their competitors, the medians of the MLEs of the parameters.

The estimation bias and mean squared error (MSEs) of each parameter are evaluated based on the obtained estimates in 10000 iteration runs. All the simulation results are reported in Tables 1 to 3. From Tables 1 to 3 we can find that the bias of MLEs is competitive with the Bayes estimates when the sample size is 10. But the MSEs of the MLEs are larger than the MSEs of the Bayes estimates. When the sample size grows, the Bayes estimates outperform the MLEs and have smaller bias and MSE for all parameters. These results indicate that the M-H MCMC method performs better than the Newton-Raphson method to obtain reliable estimates of the parameters. Please note that the largest sample size used for each stress level in simulations is 30 and the total sample size is 60. The estimation performance of the proposed M-H MCMC method is still acceptable even the sample size for ALT is not big.

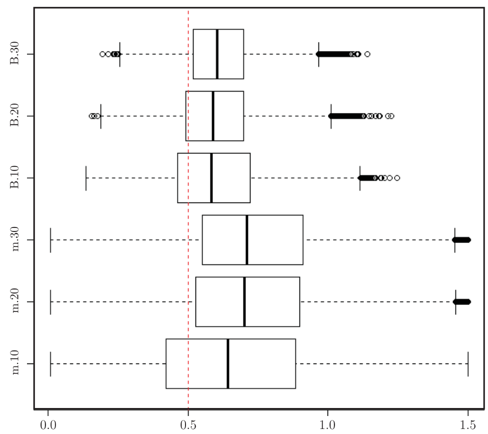

Figure 5: The box plots of 10000 estimates of b_0, in which the "m.n" and "B.n" denote the MLE and Bayes estimate of b_0, respectively, that are obtained based on samples each of size n.

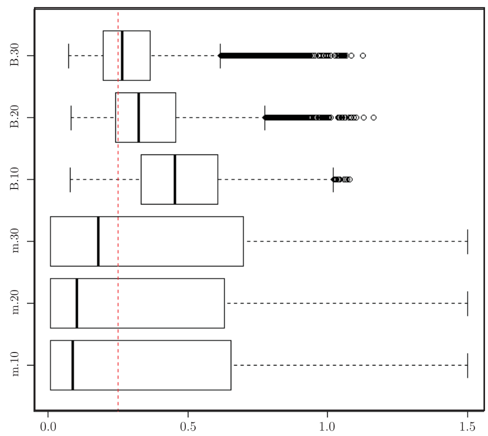

Figure 6: The box plots of 10000 estimates of b_1, in which the "m.n" and "B.n" denote the MLE and Bayes estimate of b_1, respectively, that obtained based on samples each of size n.

Table 1: The bias and MSEs of the estimates for $n = 10$.

	Bias				MSE			
	a_0	a_1	b_0	b_1	a_0	a_1	b_0	b_1
MCMC	0.1572	0.6862	0.0996	0.2341	0.296	0.7558	0.0444	0.093
MLE	0.7741	0.4973	0.1833	0.1521	1.4778	2.1301	0.1576	0.2645

Table 2: The bias and MSEs of the estimates for $n = 20$.

	Bias				MSE			
	a_0	a_1	b_0	b_1	a_0	a_1	b_0	b_1
MCMC	0.2953	0.2851	0.1015	0.1185	0.2694	0.2279	0.0342	0.0437
MLE	0.8205	0.6268	0.234	0.1504	1.3075	2.5794	0.1596	0.2402

Table 3: The bias and MSEs of the estimates for $n = 30$.

	Bias				MSE			
	a_0	a_1	b_0	b_1	a_0	a_1	b_0	b_1
MCMC	0.3717	0.1079	0.1133	0.0532	0.2686	0.1011	0.0311	0.0256
MLE	0.7534	0.8389	0.242	0.1948	1.3324	3.0301	0.1690	0.2608

It is noticed that the half-normal is the asymptotic distribution of skew-normal (SN) distribution, which contains a location parameter, a scale parameter and a skewness parameter. The model misspecification of the DTGHN and SN distributions could be a problem. However, the domain of SN is $(-\infty, \infty)$, we cannot use ALT inference method under the SN distribution for lifetime data.

5. An Example

Cooray and Anaada [4] discussed two examples regarding the stress-rupture life of Kevlar 49/epoxy, in which all lifetime components are subject to 70% and 90% stress levels, respectively, until all tested lifetime components had failed. The sample at the stress 70% contains 49 data points and the sample at the stress 90% contains 101 data points. Cooray and Anaada [4] used the GHN distribution, which has the PDF in Equation (1.1), to individually model these two data sets and confirmed that the GHN distribution is a better candidate model for these two data sets than the gamma, lognormal, Weibull and Birnbaum-Saunders distributions.

The estimation results based on the sample for the stress level 90% is more reliable due to the sample size is large and the data spread is tight than that at the stress level 70%. Hence, we consider the GHN with the MLEs as the plug-in parameters at the stress level 90% as the underlying distribution to generate ALT data sets for illustration in this section. Based on the estimation results obtained by Cooray and Anaada [4], the plug-in parameters are $\alpha_0 = 0.711$ and $\eta_0 = 1.2238$ for the GHN distribution with the PDF in Equation (3.1). We can obtain the GHN distribution with the PDF in Equation (3.2) that has the parameters $\alpha_1 = 0.711$ and $\theta_0 = \eta_0^{-2\alpha_0} = 0.7504$. Let $\mu = 0$ and $v = 5$, then we can create a DTGHN distribution for data generation. Consider the parameters in the ALT model $a_0 = 0.711$, $a_1 = 0.5$, $b_0 = 0.7504$, and $b_1 = 0.35$ for generating two data sets, each has size 50 for the normalized stress levels $s_1 = 0.5$ and $s_2 = 1$, respectively. That is, we generate 50 failure times from the DTGHN($\alpha_1 = 0.961, \theta_1 = 0.9254$) and DTGHN($\alpha_2 = 1.211, \theta_2 = 1.1004$), respectively, as the ALT samples. We would like to check the estimation performance of the proposed method based on intermediate size samples in this example. The generated data sets are given in Tables 4 and 5.

All the estimates of the ALT parameters are search over the domains $D_{a_0} = \{0.01 \leq a_0 \leq 5\}$, $D_{a_1} = \{0.01 \leq a_0 \leq 5\}$, $D_{b_0} = \{0.01 \leq b_0 \leq 5\}$ and $D_{b_1} = \{0.01 \leq b_1 \leq 5\}$, respectively. The initial solutions for implementing the maximum likelihood estimation are randomly generated from the uniform distribution over the range $(0.01, 5)$. The MLEs based on the maximum likelihood estimation are obtained by $\hat{a}_0 = 2.0815$, $\hat{a}_1 = 3.5032$, $\hat{b}_0 = 0.9665$ and $\hat{b}_1 = 1.8632$. The Bayes estimates based on the M-H MCMC method with $N = 8000$ and $M = 1600$ are obtained by $\hat{a}_{0B} = 0.8160$, $\hat{a}_{1B} = 0.3765$, $\hat{b}_{0B} = 0.6026$ and $\hat{b}_{1B} = 0.3391$. We can find that the Bayes estimates outperform the MLEs, and the Bayes estimates are closer to the true parameters than the MLEs. Hence we use the Bayes estimates to infer the quality parameter in the next step.

Table 4: 50 failure times from DTGHN($\alpha_1 = 0.961, \theta_1 = 0.9254$).

0.0262	0.0444	0.0797	0.0833	0.1161	0.1216	0.1531	0.1583	0.2428	0.3008
0.3010	0.3154	0.3354	0.3775	0.4345	0.4403	0.4649	0.5416	0.5461	0.5808
0.5980	0.6079	0.6127	0.6323	0.6663	0.6756	0.6894	0.6999	0.7589	0.7883
0.9042	0.9696	0.9928	0.9947	1.0361	1.0428	1.1464	1.3035	1.3390	1.4102
1.4320	1.4905	1.6154	1.6340	1.6454	1.6778	1.7119	1.7901	2.5964	3.1636

Table 5: 50 failure times from DTGHN($\alpha_2 = 1.211, \theta_2 = 1.1004$).

0.0540	0.0622	0.1191	0.1711	0.1731	0.2265	0.2560	0.2913	0.2997	0.3407
0.3445	0.4373	0.4484	0.4611	0.4688	0.5001	0.5177	0.5356	0.5784	0.5953
0.6046	0.7716	0.7775	0.8165	0.8183	0.8212	0.8553	0.8650	0.8728	0.8820
0.9832	0.9933	1.0001	1.0647	1.0932	1.1214	1.1393	1.1919	1.2170	1.3178
1.3736	1.4047	1.4386	1.4534	1.5078	1.6172	1.7624	2.0465	2.1307	2.5394

The Markov chains of each Bayes estimates are plotted in Figure 7. From Figure 7 we can find that the Markov chains of $\hat{a}_{0B}$ and $\hat{a}_{1B}$ are less randomized than that of the Markov chains of $\hat{b}_{0B}$ and $\hat{b}_{1B}$. But compare with the MLEs $\hat{a}_0, \hat{a}_1, \hat{b}_0$, and $\hat{b}_1$, all the Markov chains can result in better Bayes estimates to cover the true values.

Assume that we would like to study the median of lifetime components at the normal-use condition, the true median lifetime at the normal-use condition can be obtained with

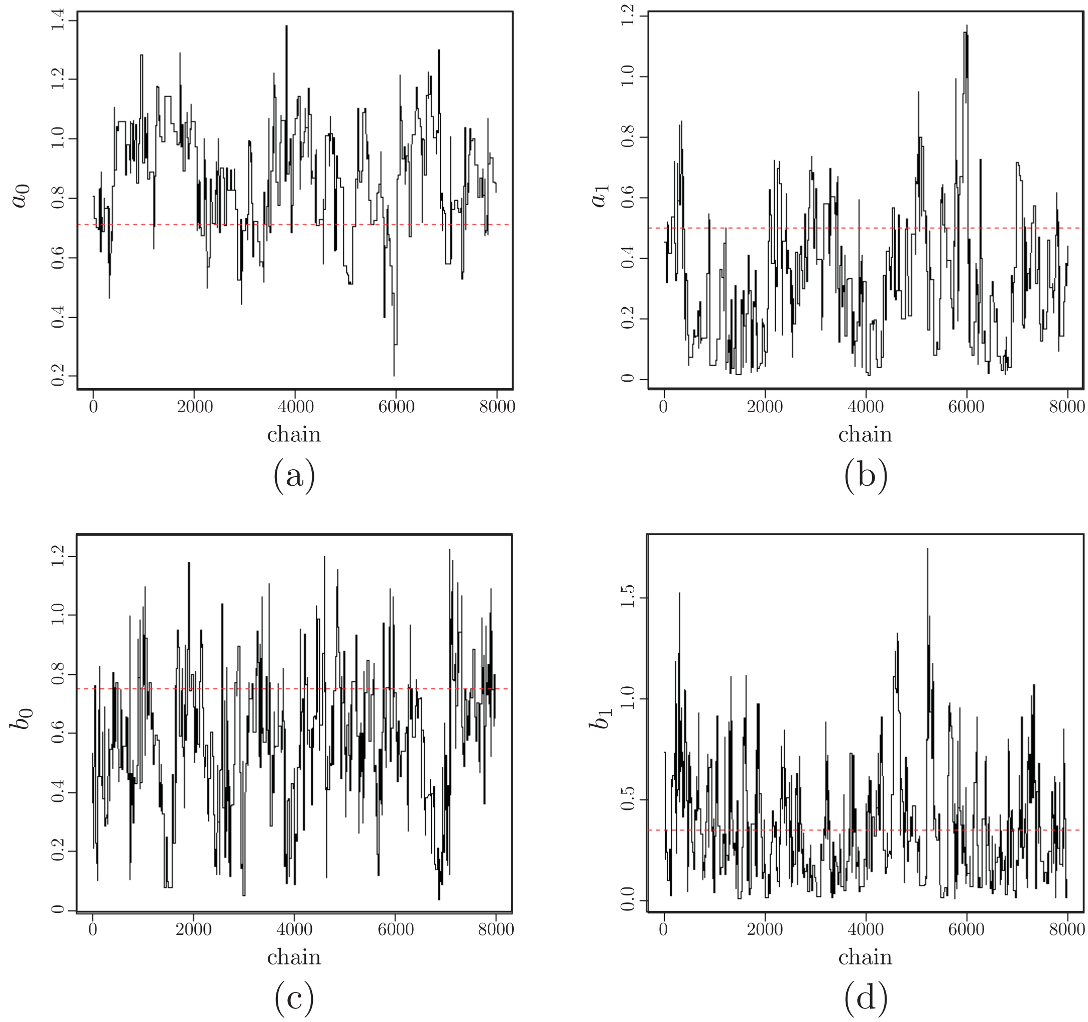

Figure 7: The Markov chains of (a) $\hat{a}_{0,B}$ (b) $\hat{a}_{1,B}$ (c) $\hat{b}_{0,B}$ and (d) $\hat{b}_{1,B}$. The dash line indicates the true parameter.

$\alpha_0 = a_0$ and $\theta_0 = b_0$ by

$$x_{0.5} = \Big(-\frac{1}{\sqrt{b_0}} \Phi^{-1}\{\Phi(-\sqrt{b_0}\mu^{a_0}) - 0.5 \times d(\mu, v; a_0, b_0)\}\Big)^{\frac{1}{a_0}} = 0.5686. \quad (5.1)$$

The MLE $\hat{x}_{0.5,B}$ through using the M-H MCMC estimates can be obtained by

$$x_{0.5B} = \Big(-\frac{1}{\sqrt{\hat{b}_{0B}}} \Phi^{-1}\{\Phi(-\sqrt{\hat{b}_{0B}}\mu^{\hat{a}_{0B}}) - 0.5 \times d(\mu, v; \hat{a}_{0B}, \hat{b}_{0B})\}\Big)^{\frac{1}{\hat{a}_{0B}}} = 0.6137. \quad (5.2)$$

Repeat the procedure to obtain the $\hat{x}_{0.5B}$ 1000 times, we can create the empirical sampling distribution of $\hat{x}_{0.5B}$ from the obtained 1000 estimates $\hat{x}_{0.5B}^{(i)}$ for $i = 1, 2, \ldots, 1000$. Then the 95% bootstrap percentile CI of $x_{0.5}$ is evaluated based on the empirical sampling distribution via using the Algorithm 2 in Section 3. The resulting CI is $(0.5480, 0.6365)$. We note that this CI covers the true value $x_{0.5} = 0.5686$. Hence, we can conclude that the median lifetime of the type product can survive at least 0.5480 unit of time to 0.6365 unit of time.

We also study the coverage probability of the proposed bootstrap percentile CI via simulations. Because the M-H MCMC method asks iterative computation for obtaining Bayes estimators and the bootstrap percentile method also asks iterative computation for obtaining the empirical sampling distribution of the estimator to construct a CI, it is very time consuming to implement simulations for obtaining the coverage probability. Figure 8 shows 150 bootstrap percentile CIs with confidence level 95%. From Figure 8 we can see that the bootstrap percentile CIs of $x_{0.5}$ are not symmetric. That is the major reason why the approximate CI of $x_{0.5}$ that is obtained via using normality approximation and Fisher information matrix are conservative. The coverage probability is 0.973 for these 150 bootstrap percentile CIs. We find that the bootstrap percentile CI of $x_{0.5}$ is also conservative but should be less conservative than that is obtained via using the normality approximation and Fisher information matrix.

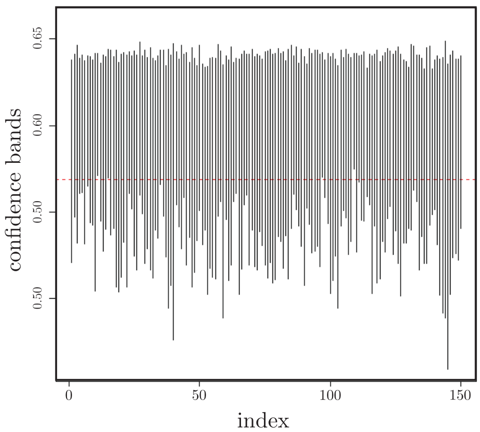

Figure 8: 150 bootstrap percentile CIs for $x_{0.5}$, which has the confidence level 95%. The dash line is the true $x_{0.5}$.

6. Conclusions

In this paper, we study the maximum likelihood estimation and Bayesian estimation methods for estimating the ALT parameters for the DTGHN distribution. The DTGHN distribution covers many lifetime distributions as special cases, so the DTGHN distribution can be a good candidate distribution for modeling lifetime data. We consider a general ALT model, in which both the shape and scale parameters are allowed to link the stress levels through using line functions. To overcome the complexity due to using the Fisher information matrix with second derivatives for implementing interval inference for the model parameters and to improve the conservative properties for the CIs that are developed based on the observed Fisher information matrix, an algorithm is proposed to obtain the bootstrap percentile CIs for the quantities of the quality characteristics of products.

The Newton-Raphson method with QN is used to implement the maximum likelihood estimation and the M-H MCMC approach is used to implement the proposed Bayesian estimation procedure. Monte Carlo simulations were conducted to evaluate the estimation performance of the proposed methods. We found that the Bayesian estimation with the M-H MCMC algorithm outperforms the maximum likelihood estimation with the Newton-Raphson method even based on intermediate size samples. Roughly, the Bayes estimates have smaller bias and MSE than that of the MLEs. Because we use non-informative prior distributions for the model parameters to develop the M-H MCMC method, the Bayes estimates are close to the MLEs.

An example regarding the stress-rupture life of Kevlar 49/epoxy is used to illustrate the applications of the proposed methods. In the illustrative example, we find that there are room to improve the estimation for the parameters a_0 and a_1 via using the proposed M-H MCMC method. Because we use non-informative prior distributions to characterize the model parameters and use uniform distributions as proposals, the convergence of the proposed M-H MCMC method is slow. How to improve the estimation performance of the proposed Bayesian method and bootstrap CI inference procedure for quantiles are two interesting topics. The half-normal is the asymptotic distribution of SN distribution, which contains a location parameter, a scale parameter and a skewness parameter. It is known that the SN distribution contains the normal distribution as special case. But the SN distribution cannot be used for the reliability applications based on lifetime data due to the domain of the SN distribution can be negative. Hence, how to define a generalized SN (GSN) distribution and a truncated GSN (TGSN) distribution and extend the proposed method for a TGSN distribution also are three important topics. All these five topics will be studied in the future.

Acknowledgements

This study is jointly supported by the grants of Heilongjiang Bureau of Statistics, China NO:2018B11.

References

[1] Ahmadi, K., Rezaei, M. and Yousefzadeh, F. (2015). *Estimation for the generalized half-normal distribution based on progressive Type-II censoring*, Journal of Statistical Computation and Simulation, Vol.85, 1128-1150.

[2] Altun, E., Yousof H. M. and Hamedani, G. G. (2018). *A new generalization of generalized half-normal distribution: properties and regression models*, Journal of Statistical Distributions and Applications, 5:7; https://doi.org/s40488-018-0089-4.

[3] Byrd, R. H., Lu, P., Nocedal, J. and Zhu, C. (1995). *A limited memory algorithm for bound constrained optimization*, SIAM Journal of Scientific Computing, Vol.16, 1190-1208.

[4] Cooray, K. and Ananda, M. M. (2008). *A generalization of the half-normal distribution with applications to lifetime data*, Communications in Statistics-Theory and Methods, Vol.37, 1323-1337.

[5] Cordeiro, G. M., Pescim, R. R. and Ortega, E. M. (2012). *The Kumaraswamy generalized half-normal distribution for skewed positive data*, Journal of Data Science, Vol.10, 195-224.

[6] Cordeiro, G. M., Alizadeh, M., Pescim, R. R. and Ortega, E. M. (2017). *The odd log-logistic generalized half-normal lifetime distribution: properties and applications*, Communications in Statistics-Theory and Methods, Vol.46, 4195-4214.

[7] He, Q., Zha, Y., Sun, Q., Pan, Z. and Liu, T. (2017). *Capacity fast prediction and residual useful life estimation of valve regulated lead acid battery*, Mathematical Problem in Engineering, Article ID 7835049, 9 pages.

[8] Lawless, J. F. (2003). Statistical Models and Methods for Lifetime Data, John Wiley & Sons, NY, USA.

[9] Meeker, W. Q. and Escobar, L. A. (1998). Statistical Methods for Reliability Data, John Wiley & Sons, NY, USA.

[10] Nelson, W. B. (1990). Accelerated Testing: Statistical Models, Test Plans and Data Analysis, John Wiley & Sons, NJ, USA.

[11] Nogales, A. G. and Pérez, P. (2015). *Unbiased estimation for the general half-normal distribution*, Communications in Statistics-Theory and Methods, Vol.44, 3658-3667.

[12] Olmos, N. M., Varela, H., Bolfarine, H. and Gómez, H. W. (2014). *An extension of the generalized half-normal distribution*, Statistical Papers, Vol.55, 967-981.

[13] Pescim, R. R., Ortega, E. M., Cordeiro, G. M., Alizadeh, M. (2017). *A new log-location regression model: estimation, influence diagnostics and residual analysis*, Journal of Applied Statistics, Vol.44, 233-252.

[14] Pewsey, A. (2002). *Large-sample inference for the general half-normal distribution*, Communications in Statistics-Theory and Methods, Vol.31, 1045-1054.

[15] Pewsey, A. (2004). *Improved likelihood based inference for the general half-normal distribution*, Communications in Statistics-Theory and Methods, Vol.33, 197-204.

[16] Powell, P. C. (1983). Engineering with Polymers, New York: Chapman and Hall.

[17] Srivastava, P. W. and Savita, S. (2018). *An accelerated life test plan for a two-component parallel system under ramp-stress loading using masked data*, International Journal of Quality and Reliability Management, Vol.35, 811-820.

[18] Wachtman, J. B. (1996). Mechanical Properties of Ceramics, New York: John Wiley & Sons, Inc, Vol.69, 265-267.

[19] Wang, W. and Kececioglu, D. B. (2000). *Fitting the Weibull log-linear model to accelerated life-test data*, IEEE Transactions on Reliability, Vol.49, 217-223.

[20] Wang, G., Niu, Z. and He, Z. (2015). *Accelerated lifetime data analysis with a nonconstant shape parameter*, Mathematical Problem in Engineering, Article ID 801465, 8 pages.

[21] Wang, L. (2018). *Estimation for the parameters of the generalized half-normal distribution*, Journal of Statistical Computation and Simulation, Vol.88, 629-645.

School of Mathematics and Statistics, Northeast Petroleum University, Daqing, Heilongjiang, CHN.

E-mail: xinhuaan@nepu.edu.cn

Major area(s): Big data and applied statistics.

Department of Management and Data-Mining Research Center, Xiamen University, Xiamen, Fujian, CHN.

E-mail: xmjpzhu@xmu.edu.cn

Major area(s): Data science and business intelligence, applications of mathematical statistics and econometric model and applications.

(Received October 2018; accepted December 2019)