International Journal of
Information
and
Management
Sciences

International Journal of Information and Management Sciences
29 (2018), 115-125. DOI:10.6186/IJIMS.2018.29.1.5

Minimax Distribution Free Procedure for Inventory Models

Han-Wen Tuan

Hungkuang University

Abstract

To shed more light into using the minimax distribution free procedure to obtain the optimal order quantity, this paper specifically focuses on the work of Gallego (1992). The previous author created a two-point distribution to serve as the most unfavorable case among the distribution with the same mean and variance to estimate the expected cost of the lost sales. However, such solution procedure contains a sequence that is dependent on whether it is convergent or not. The purpose of this paper is three fold. First is to point out some questionable areas in the solution procedure of Gallego (1992). Second is to find a reasonable condition to ensure the existence of the optimal order quantity. Lastly, numerical examples are given to illustrate the findings.

Keywords: Inventory models, minimax distribution-free procedure.

1. Introduction

The lead time demand for inventory models is usually assumed to follow the normal distribution, but it cannot provide a proper approximation for other distribution types. On the other hand, actual business demand is often difficult to forecast such that researchers are unable to accurately estimate its distribution. Nonetheless, as researchers can evaluate expect value and variance using historical information, the two-point distribution proposed by Gallego [1] becomes a possible tool to stand for the worse-case scenario. The savings from the normal distribution to the worst case can be applied to find the actual distribution of the lead time demand. The first paper to consider the minimax distribution free approach to derive the optimal order quantity was in 1958 by Scarf [11]. This method only used the mean and the variance of demand without any assumptions on the type of demand distribution. It was not until Gallego [1] who applied the same approach for an inventory model with backorder which was considered a breakthrough in the estimation of lead time demand. He developed a two-point distribution to serve as the worst case among all distribution types with the same expected value and standard derivation to evaluate cost of lost sales. It involves the continuous inventory model with a mixture of backorders and lost sales. However, the existence and uniqueness of the solution using such method has not been verified. Under a reasonable condition, this paper intends to provide a strong foundation to support it. Many papers had adopted such method in lead time demand studies for the same purpose as well as to obtain optimal order quantity and optimal lead time in the worst possible case. For example, Gallego and Moon [2] further extended minimax distribution free approach to deal with the newsboy problem; Ouyang and Chang [7] considered the mixed inventory model; Ouyang and Chuang [8] studied the inventory model under service levels constraint; and Wu et al. [13] considered the inventory models with controllable lead time. Hence, it is possible to establish that this approach has become an important method when dealing with inventory models of unknown demand distribution. Up to now, there is a total of 52 papers that had cited Gallego [1] in their references. However, only Lin et al. [5] provided a solution method to replace the iterative sequence proposed by Gallego [1] without discussing the latters sequence approach. Other recent papers that applied the minimax distribution free approach include: Hung et al. [3], Lin and Chu [6], Kumar and Goswami [4], Qi et al. [9], Şen and Talebian [10], Wright [12], and Wu and Warsing [14]. The rest of the other citations only mentioned the work of Gallego [1] in their introduction. This study explores the solution of Gallego [1] in detail to provide a more valid support for future scholars who want to continue working on this method. The rest of the paper is organized as follows: in the first part, a proof is shown that the sequence will converge and the limit is a feasible solution of Gallego [1]. In the later part, a method to find the criterion to guarantee the sequence will converge and that the limit is becomes a feasible solution is presented.

2. Notation

For consistency with previous studies, the same notation and the assumptions are used as in Gallego [1] and Gallego and Moon [2].

D = average demand per year.
F = the cumulative distribution of the lead time demand.
h = inventory carrying cost per item per year.
K = the fixed ordering cost per order.
Q = order quantities per order.
R = reorder point, under the assumption $R \geq \mu$.
π = unit shortage cost.
μ = mean lead time demand.
$\Delta = R - \mu$.

σ^2 = variance of the lead time demand.

$E(x-R)^+ = \int \max(x-R,0)dF(x)$ is the expected shortage units per inventory cycle.

Q_1 = the economic order quantity with $Q_1 = \sqrt{\frac{2KD}{h}}$.

M = the maximum feasible solution for the order quantity with $M = \sqrt{\frac{2KD+\pi\sigma D}{h}}$.

3. Review of Results

Gallego [1] considered the average annual total cost for inventory model with lead time demand distribution F as

$$C(Q,R,F) = \frac{KD}{Q} + h\Big(\frac{Q}{2} + R - \mu\Big) + \frac{\pi D}{Q}\int \max(x-R,0)dF(x). \tag{3.1}$$

F has known finite first and second moments but there is no assumption on the distribution form of F. Gallego used Ω to denote the class of cumulative distributions function with mean μ and variance σ^2. He applied the minimax approach of Scarf [11] to find the most unfavorable distribution in Ω for each (Q,R) and then minimized over (Q,R). Gallego derived the following upper bound to estimate the expected shortage units per cycle.

Proposition 1 (Gallego [1]). *The least favorable distribution $F \in \Omega$ is a two point distribution. Moreover,*

$$\max_{F\in\Omega}\int \max(x-R,0)dF(x) = \frac{\sqrt{\sigma^2+\Delta^2}-\Delta}{2}.$$

Many papers have used the upper bound in Proposition 1 of Gallego [1]. The following minimum problem for the expected annual total cost was considered:

$$C(Q,\Delta) = \frac{KD}{Q} + h\Big(\frac{Q}{2} + \Delta\Big) + \frac{\pi D}{2Q}(\sqrt{\Delta^2+\sigma^2}-\Delta). \tag{3.2}$$

over $\{(Q,\Delta) : Q > 0, \Delta \geq 0\}$. It was further derived that $\frac{\partial C(Q,\Delta)}{\partial Q}$ and $\frac{\partial C(Q,\Delta)}{\partial \Delta}$, then a solution was made for $\frac{\partial C(Q,\Delta)}{\partial Q} = 0$ and $\frac{\partial C(Q,\Delta)}{\partial \Delta} = 0$ under the restriction $\Delta \geq 0$ and $Q > 0$. From $\frac{\partial C(Q,\Delta)}{\partial Q} = 0$, the following equation was obtained

$$Q = \sqrt{\frac{2KD}{Q} + \frac{\pi D}{h}(\sqrt{\Delta^2+\sigma^2}-\Delta)}. \tag{3.3}$$

From $\frac{\partial C(Q,\Delta)}{\partial \Delta} = 0$, it was found that

$$\frac{\Delta}{\sqrt{\Delta^2+\sigma^2}} \geq 1 - \frac{2hQ}{\pi D}. \tag{3.4}$$

under the condition $\Delta \geq 0$, and finally discovered the restriction for the feasible solution, which is

$$\pi D \geq 2hQ. \tag{3.5}$$

Without detailed explanation, Gallego claimed [1] that under the restriction $\pi D \geq 2hQ$,

$$\Delta = \frac{(\pi D - 2hQ)\sigma}{2\sqrt{(hQ(\pi D - hQ))}} \tag{3.6}$$

and

$$\sqrt{\Delta^2 + \sigma^2} - \Delta = \sigma\sqrt{\frac{hQ}{\pi D - hQ}}. \tag{3.7}$$

Substituting Equations (3.5) and (3.6), the following equation is obtained

$$Q = \sqrt{\frac{2KD}{h} + \frac{\pi D\sigma}{h}\sqrt{\frac{hQ}{\pi D - hQ}}}. \tag{3.8}$$

One of the main assumption is that $Q_1 = \sqrt{\frac{2KD}{h}}$ and if the restriction in Equation (3.5) was satisfied, Equation (3.8) can be used to update Q. It was known that the sequence of updates is an increasing sequence, say (Q_n). The procedure was repeated until it was convergent or until the restriction in Equation (3.5) was violated. If the sequence converged to Q^*, then Equation (3.6) was used to find Δ^* to imply that (Q^*, Δ^*) satisfied the first order conditions and the restriction in Equation (3.5) was satisfied, hence; the pair (Q^*, Δ^*) is optimal.

Otherwise, if the restriction in Equation (3.5) was violated, then the following pair

$$(Q^*, \Delta^*) = \left(\sqrt{\frac{2KD + \pi D\sigma}{h}}, 0\right) \tag{3.9}$$

is claimed to be the optimal solution.

The economic order quantity, denoted as Q_1, was considered the starting point to run the iterative process in generating a sequence, say (Q_n) such that the convergent solution was found. From Equation (3.8), it is easy to show that the sequence (Q_n) is an increasing sequence. This paper implies that if (Q_n) does has an upper bound then it will converge to its least upper bound. On the other hand, if (Q_n) has no upper bound, then it will diverge to ∞. In this case, Gallego [3] failed to consider the inventory policy when the sequence diverges to ∞.

In the following, we point out some questionable results in Equation (3.7) such that if we follow Gallego [1], then we may divide it into two cases: (1) $\pi D = 2hQ$ and (3.2) $\pi D > 2hQ$. Alternatively, we derive a procedure to avoid the iterative algorithm of Gallego. We first find a reasonable restriction (that is $Q_1 \equiv \sqrt{2DK/h} < Q < \sqrt{(2K + \pi\sigma)D/h} \equiv M$) and then under this reasonable restriction, we directly prove that there is a unique optimal solution of Q that satisfies the Equation (3.8) (that is Equation (4.10)).

4. Improved Solution Procedure for the Inventory Model

Under the restriction $\pi D \geq 2hQ$, $\dfrac{\partial C(Q,\Delta)}{\partial \Delta}=0$ yields that $\dfrac{\Delta}{\sqrt{\Delta^2+\sigma^2}}=\dfrac{\pi D-2hQ}{\pi D}$. After squaring both sides and applying cross-multiplication, it shows

$$\frac{\Delta^2}{\sigma^2}=\frac{(\pi D-2hQ)^2}{(\pi D)^2-(\pi D-2hQ)^2}=\frac{(\pi D-2hQ)^2}{4hQ(\pi D-2hQ)}. \tag{4.1}$$

From Equation (4.1), it is easy to imply Equation (3.6). Using $\dfrac{\Delta}{\sqrt{\Delta^2+\sigma^2}}=\dfrac{\pi D-2hQ}{\pi D}$ again, it can be obtained that $\pi D(\sqrt{\Delta^2+\sigma^2}-\Delta)=2hQ\sqrt{\Delta^2+\sigma^2}$, which then yields

$$\begin{aligned}\sqrt{\Delta^2+\sigma^2}-\Delta&=\frac{2hQ}{\pi D}\sqrt{\Delta^2+\sigma^2}=\frac{2hQ}{\pi D}\frac{\pi D}{\pi D-2hQ}\Delta\\&=\frac{2hQ}{\pi D}\frac{\pi D}{\pi D-2hQ}\frac{(\pi D-2hQ)\sigma}{2\sqrt{(hQ(\pi D-hQ))}}.\end{aligned} \tag{4.2}$$

After carefully examining the previous results, it is not possible to Equation (3.7) from Equation (4.1) under the restriction $\pi D \geq 2hQ$. There has to be a stronger condition such as $\pi D > 2hQ$, to be able to derive Equation (3.7). With this discovery, this paper considered an alternative method to derive the optimal solution while avoiding the convergent problem for the sequence in Gallego [1]. A possible range for Q was determined and if this is considered as a function of Δ, then the following equation can be derived from Equation (3.3):

$$Q(\Delta)=\sqrt{\frac{2KD}{h}+\frac{\pi D}{h}(\sqrt{\Delta^2+\sigma^2}-\Delta)}. \tag{4.3}$$

The next section provides an alternative method to derive Equation (3.8), without referring to equation (3.7).

From $\dfrac{\partial C(Q,\Delta)}{\partial Q}=0$ and $\dfrac{\partial C(Q,\Delta)}{\partial \Delta}=0$, it is possible to derive an equation that contains only one variable Q. Using $\dfrac{\partial C(Q,\Delta)}{\partial Q}=0$, it is known that

$$\left(\frac{h}{D}Q^2-2k\right)\frac{1}{\pi}=\sqrt{\Delta^2+\sigma^2}-\Delta. \tag{4.4}$$

By $\dfrac{\partial C(Q,\Delta)}{\partial Q}=0$, it yields that

$$\frac{2hQ}{\pi D}=1-\frac{\Delta}{\sqrt{\Delta^2+\sigma^2}}=\frac{\sqrt{\Delta^2+\sigma^2}-\Delta}{\sqrt{\Delta^2+\sigma^2}} \tag{4.5}$$

Hence, $\dfrac{2hQ}{\pi D}=\dfrac{\frac{h}{D}Q^2-2k}{\pi\sqrt{\Delta^2+\sigma^2}}$, and show that

$$2hQ\sqrt{\Delta^2+\sigma^2}=hQ^2-2DK, \tag{4.6}$$

by substituting Equation (4.4) into Equation (4.6), it can be implied that

$$2hQ\Big[\Big(\frac{h}{D}Q^2 - 2k\Big)\frac{1}{\pi}\Big] = hQ^2 - 2DK. \tag{4.7}$$

Subsequently, plugging Equation (3.6) into Equation (4.7) yields

$$2hQ(hQ^2 - 2KD) + \frac{\pi DQ\sigma(\pi D - 2hQ)}{\sqrt{hQ(\pi D - hQ)}} = hQ^2 - 2KD. \tag{4.8}$$

To present a more compact form, Equation (4.8) is rewritten as

$$\frac{\pi DhQ\sigma(\pi D - 2hQ)}{\sqrt{hQ(\pi D - hQ)}} = (hQ^2 - 2KD)(\pi D - 2hQ). \tag{4.9}$$

Under the stronger condition of $\pi D > 2hQ$ for Q in some feasible range that will be explained later, $\pi D - 2hQ$ can be cancelled out from both sides of Equation (4.9). Therefore, the solution for the system of the first partial derivatives is found as

$$\pi D\sigma\sqrt{\frac{hQ}{\pi D - hQ}} + 2KD = hQ^2. \tag{4.10}$$

Equation (4.10) proves that it is possible to derive the result of Equation (3.8) without referring to Equation (3.7).

The next part explains the definition of a feasible range for the order quantity. Since the feasible range for Δ is $0 \le \Delta < \infty$, from $\sqrt{\Delta^2 + \sigma^2} - \Delta = \dfrac{\sigma^2}{\sqrt{\Delta^2 + \sigma^2} + \Delta}$, it is known that $Q(\Delta)$ is a decreasing function of Δ with its maximum value of $Q(0) = \sqrt{\dfrac{2KD + \pi D\sigma}{h}} = M$ and its inferior value of $Q_1 = \sqrt{\dfrac{2KD}{h}}$. This finding is summarized in the following lemmas.

Lemma 1. *Under the assumption that* $Q(\Delta) = \sqrt{\dfrac{2KD}{h} + \dfrac{\pi D}{h}(\sqrt{\Delta^2 + \sigma^2} - \Delta)}$ *for* $0 \le \Delta < \infty$*, the maximum value is* $Q(0) = \sqrt{\dfrac{2KD + \pi D\sigma}{h}} = M$ *and the inferior value is* $\sqrt{\dfrac{2KD}{h}} = Q_1$.

Proof. $Q(\Delta)$ is defined in Equation (4.3) it is seen that $Q(\Delta)$ decreases with respect to Δ so the maximum value is $Q(0)$ and the minimum value is

$$\lim_{\Delta\to\infty} Q(\Delta) = \lim_{\Delta\to\infty}\sqrt{\Big[\frac{2KD}{h} + \frac{\pi D}{h}\Big(\frac{\sigma^2}{\sqrt{\Delta^2 + \sigma^2} + \Delta}\Big)\Big]} = \sqrt{\frac{2KD}{h}}. \tag{4.11}$$

Through Lemma 1, there is a sufficient criterion to guarantee that the solution for the system of the first partial derivatives will satisfy the restriction in Equation (3.5).

Lemma 2. *If the criterion $2hM \leq \pi D$ holds, then the maximum value of $Q(\Delta)$ from Equation (4.3) satisfies Equation (3.5).*

Proof. After deriving that $Q(\Delta) \leq Q(0) = M$ and under the condition of $2hM \leq \pi D$, $2hQ(\Delta) \leq \pi D$ is obtained, which is Equation (3.5) as proposed by Gallego [1].

Due to technical reasons, there has to be a criterion stronger than $2hM \leq \pi D$. Therefore, the following observation is noted.

Observation 1. *It can be claimed that $2hM < \pi D$ from a practical point of view. (Numerical examination) Using the numerical examples of Gallego [1] the values of $\dfrac{2hM}{\pi D}$ yield 0.16 and 0.13 indicating that $2hM < \pi D$ is supported. From a practical point of view, Observation 1 may be valid.*

To ensure reliability, a sensitivity analysis was done on Observation 1. Only one parameter had varying values either increasing or decreasing up to 50%, for parameters: K, σ, D and h. The original results of $2hM/\pi D = 0.16$ and 0.13 are already known based on Examples 1 and 2 of Gallego [1]. This paper proposes to rewrite as

$$\frac{2hM}{\pi D} = \frac{2}{\pi}\sqrt{\frac{(2K + \pi\sigma)h}{D}} \tag{4.12}$$

to imply that the results of sensitivity analysis of $2hM/\pi D$ will be within the interval of $[0.065, 0.320]$, where $0.065 = (0.13)0.5$ and $0.320 = 0.16/0.5$. After doing sensitivity analysis, it is further proven that $2hM/\pi D < 1$.

In the next sections, Observation 1 is assumed to hold true throughout the paper. The results are summarized in the next lemma.

Lemma 3. *Under the assumption of $2hM < \pi D$, for order quantity Q between $\sqrt{\dfrac{2KD}{h}} = Q_1$ and $\sqrt{\dfrac{2KD + \pi D\sigma}{h}} = M$, we have $0 < \pi D - 2hQ$.*

Proof. Observation 1 already shows that $2hM < \pi D$. Lemma 1 provides that $\sqrt{\dfrac{2KD}{h}} = Q_1 < Q(\Delta) \leq Q(0) = M$. Hence, it can be derived that $2hQ < \pi D$.

Motivated by Equation (4.10), solving Equation (4.10) is equivalent to solving $f(Q) = 0$, with

$$f(Q) = (hQ^2 - 2DK)^2(\pi D - hQ) - h\pi^2\sigma^2 D^2 Q. \tag{4.13}$$

Hence, it is derived that

$$f'(Q) = 4hQ(hQ^2 - 2DK)(\pi D - hQ) - h(hQ^2 - 2DK)^2 - h\pi^2\sigma^2 D^2. \tag{4.14}$$

and subsequently

$$f''(Q) = 4h(hQ^2 - 2DK)(\pi D - 3hQ) + 8h^2Q^2(\pi D - hQ)^2. \tag{4.15}$$

Equation (4.13) can then be rewritten as

$$f''(Q) = 4h(3hQ^2 - 2DK)(\pi D - 2hQ) + 4h^2Q(hQ^2 + 2DK).$$

Using Lemma 3, $f''(Q) > 0$ for $Q \in (Q_1, M)$ can be obtained. Moreover, it is known that $f(Q_1) = -h\pi^2\sigma^2D^2Q_1 < 0$ and $f(M) = (\pi\sigma D)^2(\pi D - 2hM) > 0$. On the other hand, it implies that $f'(Q_1) = -h\pi^2\sigma^2D^2 < 0$ and

$$f'(M) = 4h\pi\sigma DM(\pi D - 2hM) + 4h^2\pi\sigma DM^2 - 2h(\pi\sigma D)^2. \tag{4.16}$$

Since, $hM^2 = 2AD + \pi\sigma D > \pi\sigma D$, it can be implied that $f'(M) > 0$. Combining these results to conclude that $f(Q)$ is convex for $Q \in [Q_1, M]$ with $f'(Q_1) < 0$ and $f'(M) > 0$ to imply that there is a point, denoted as $Q_{\min}$ that satisfies $f'(Q_{\min}) = 0$ such that $f(Q)$ decreases for $Q \in [Q_1, Q_{\min}]$ and $f(Q)$ increases for $Q \in [Q_{\min}, M]$. It further yields $Q_{\min}$ where $Q_{\min}$ is the minimum for $f(Q)$ with $Q_1 \le Q \le M$. From $f(Q_1) < 0$ and $f(M) > 0$, it shows that there exists a unique point, say Q^*, with $f(Q^*) = 0$. A summary of these findings is presented in the next theorem.

Theorem 1. *Under the constraint $\pi D > 2hM$, we prove that there is a unique Q^* in $[Q_1, M]$ that satisfies the system of the first partial derivatives.*

Proof. $f(Q)$ is derived to be a convex function with minimum point $Q_{\min}$. Using $f(Q_1) < 0$ and $f(M) > 0$ to yield $f(Q_{\min}) < 0$ and $f(M) > 0$ with $f(Q)$ increases for $Q \in [Q_{\min}, M]$. There is indeed a unique point, say Q^*, that satisfies $Q_{\min} < Q^* < M$ and $f(Q^*) = 0$.

As previously mentioned, solving Equation (4.10) is equivalent to solving $f(Q) = 0$. Equation (4.10) is proven to have a unique solution Q^*. Hence, (Q^*, Δ^*) is the unique solution for the first partial system, where Δ^* is derived by Equation (4.1).

Corollary 1. *This paper proves that the optimal solution of Δ, denoted as Δ^* exists and is unique.*

Proof. Based on Equation (4.1), after obtaining the optimal solution of Q^*, then

$$\frac{\Delta^*}{\sigma} = \sqrt{\frac{\pi D - 2hQ^*}{4hQ^*}} \tag{4.17}$$

that it is exists and is unique.

The results above prove there is a unique solution and as such, it is possible to avoid the tedious procedure in proposed by Gallego [1] and can instead use a better procedure to solve the convergent problem. Moreover, it is shown that under the observation $\pi D > 2hM$, the degenerated case mentioned in Gallego [1] with $Q^* = M$ and $\Delta^* = 0$ will not happen.

For completeness, the following algorithm is provided to help researchers in finding optimal solutions:

Step 1. Solve $f(Q) = 0$, where $f(Q)$ is Equation (4.11), and then denote the root as Q^*.

Step 2. Solve Δ^* by Equation (4.15).

Step 3. Solve $C(Q^*, \Delta^*)$ by plugging Q^* and Δ^* in Equation (3.2).

5. Numerical Examples

To illustrate the proposed procedure, the same numerical examples in Gallego [1] are used. In Example 1, $K = 70$, $D = 10,000$, $h = 0.6$, $\mu = 300$, $\sigma = 40$, and $\pi = 1.5$. With the starting point, $Q_1\sqrt{\frac{2KD}{h}}$, executing the iterative algorithm of Gallego to derive the sequence as is shown in Table 1.

Table 1: For Example 1 by Gallego [1] to derive the sequence.

Sequence	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
Order quantities Q	1527.525	1608.862	1611.085	1611.145	1611.147	1611.147

Hence, after six steps of mathematical calculations, the optimal order quantity is $Q^* = 1611.147$. The next step is to use the numerical analysis method with initial value $Q = 1,600$. Since $\sqrt{\frac{2KD}{h}} = 1527.525$ and $\sqrt{\frac{2KD + \pi D\sigma}{h}} = M = 1825.742$, it is possible to directly solve the root for Equation (4.11), which yields 1611.147 as compared to $Q^* = 1594$ in Gallego [1].

In Example 2, $K = 3.2$, $D = 220$, $h = 2.88$, $\mu = 30$, $\sigma = 10.5$, and $\pi = 32$. With the starting point, $Q_1 = \sqrt{\frac{2KD}{h}}$, the iterative algorithm of Gallego [1] is then executed to derive the sequence as follows.

Table 2: For Example 1 by Gallego [1] to derive the sequence.

Sequence	Q_1	Q_2	Q_3	Q_4	Q_5
Order quantities Q	22.111	54.232	65.993	69.023	69.743
Sequence	Q_6	Q_7	Q_8	Q_9	Q_{10}
Order quantities Q	69.911	69.950	69.959	69.961	69.961

Hence, after 10 steps of mathematical calculations, the optimal order quantity is $Q^* = 69.961$. The next step is to use the numerical analysis method in the proposed procedure with initial value $Q = 100$. Since $\sqrt{\frac{2KD}{h}} = 22.111$ and $\sqrt{\frac{2KD + \pi D\sigma}{h}} = M = 161.727$, a direct solution for finding the root of Equation (4.11) yields 69.961 as opposed to $Q^* = 69$ in Gallego [1]. Based on two numerical examples above, it is evident that the procedure can efficiently derive the optimal order quantity and thus making the tedious iterative procedure of Gallego [1] to generate a convergent sequence unnecessary.

6. Direction for Future Research

One possible direction for future research is to provide a complete solution structure that may contain the following two cases:

Case (i), when $\pi D \le 2hQ$, then the optimal solution occurs on the boundary with $\Delta^* = 0$ that is a patch work for what Gallego [1] overlooked: $\pi D < 2hQ$.

Case (ii), when $\pi D > 2hQ$, an analytical proof to show that the sequence generated by Gallego [1] is convergent.

7. Conclusion

There are different demand types even for the same products because market situations are different and it is not easy to forecast the demand accurately. In order to keep the service level, it becomes an important issue for decision makers to consider the safe level of stocks that need to be prepared. Many scholars use minimax distribution-free procedure to explore the varying demand and derive the optimal replenishment policy. This study determined the criteria to guarantee that the limit of a convergent sequence is a feasible solution. The solutions are illustrated by solving some examples, indicating the accuracy and completeness of our procedure. Lastly, it is considered that

$$\lim_{\Delta \to \infty} Q(\Delta) < Q < \min\{Q(0), \pi D/2h\} \tag{7.1}$$

will be an interesting research topic in the future.

Acknowledgements

The author greatly appreciated the financial support from Hungkuang University and the English revisions by Kaye Lee (email: kayetanlee@gmail.com).

References

[1] Gallego, G. (1992). *A minimax distribution-free procedure for the* (Q, R) *inventory model*, Operations Research Letters, Vol.11, 55-60.

[2] Gallego, G. and Moon, I. (1993). *The Distribution-free newsboy problem- review and extensions*, Journal of the Operational Research Society, Vol.44, 825-834.

[3] Hung, C., Hung, K., Tang, W, Lin, R. and Wang, C. (2009). *Periodic review stochastic inventory models with service level constraint*, International Journal of Systems Science, Vol.40, No.3, 237-243.

[4] Kumar, R. S. and Goswami, A. (2015). *A continuous review production-inventory system in fuzzy random environment: Minmax distribution free procedure*, Computers and Industrial Engineering, Vol.79, 65-75. doi:10.1016/j.cie.2014.10.022.

[5] Lin, R., Chouhuang, W. T., Yang, G. K. and Tung, C. (2007). *An improved algorithm for the minimax distribution-free inventory model with incident-oriented shortage costs*, Operations Research Letters, Vol.35, No.2, 232-234.

[6] Lin, R. H. and Chu, P. (2006). *Note on stochastic inventory models with service level constraint*, Journal of the Operations Research Society of Japan, Vol.49, No.2, 117-129.

[7] Ouyang, L. Y. and Chang, H. C., (2002). *A minimax distribution free procedure for mixed inventory models involving variable lead time with fuzzy lost sales*, International Journal of Production Economics, Vol.76, 1-12.

[8] Ouyang, L. Y. and Chuang, B. R. (2000). *A periodic review inventory model involving variable lead time with a service level constraint*, International Journal of Systems Science, Vol.31, 1209-1215.

[9] Qi, A., Ahn, H. and Sinha, A. (2017). *Capacity investment with demand learning*, Operations Research, Vol.65, No.1, 145-164. doi:10.1287/opre.2016.1561.

[10] Şen, A. and Talebian, M. (2017). *Markdown budgets for retail buyers: Help or hindrance*? Production and Operations Management, Vol.26, No.10, 1875-1892. doi:10.1111/poms.12725

[11] Scarf, H. (1958). A min-max solution of an inventory problem, Studies in The Mathematical Theory of Inventory and Production (Chapter 12) Stanford Univ. Press., Stanford, CA, 1958.

[12] Wright, C. P. (2014). *Decomposing airline alliances: A bid-price approach to revenue management with incomplete information sharing*, Journal of Revenue and Pricing Management, Vol.13, No.3, 164-182. doi:10.1057/rpm.2013.46.

[13] Wu, J. W., Lee, W. C. and Tsai, H. Y. (2002). *A note on minimax mixture of distributions free procedure for inventory model with variable lead time*, Quality & Quantity, Vol.36, 311-323.

[14] Wu, X. and Warsing Jr., D. P. (2013). *Comparing traditional and fuzzy-set solutions to* (Q, r) *inventory systems with discrete lead-time distributions*, Journal of Intelligent and Fuzzy Systems, Vol.24, No.1, 93-104. doi:10.3233/IFS-2012-0533.

Department of Computer Science and Information Management, Hungkuang University, Taiwan.
E-mail: dancathy@hk.edu.tw
Major area(s): Pattern recognition, operation research.

(Received May 2017; accepted December 2017)