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Note on Inventory Model with Reorder Point as a Decision Variable

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Abstract

We examine Horowitz and Daganzo (1986) to provide a patch work such that researchers can realize their paper as the first article to treat the reorder point as a decision variable for inventory models with stochastic lead time. However, only seven papers had cited their papers in their references. We may claim that (a) Their derivations contains tedious verification, (b) They adopted a graphical method only search less than fifty percent of the possible domain and (c) They did not show that within their shrunk sub-domain, the existence and uniqueness of their approximated optimal solution that result in these few citations. Our improvements include (i) A clear derivation after they reduced to one variable problem, (ii) An analytical procedure to shrink the searching domain to 91%, and (iii) Within our shrunk sub-domain, we prove the existence and uniqueness of our exact optimal solution.

Keywords: Reordered point, safety factor, inventory.

1. Introduction

In this paper, we study the inventory model with expedited shipment for shortage where the reorder point and the order quantity are decision variables. This plan was based on the paper of Horowitz and Daganzo [9] that was published on Production and Inventory Management It is the first paper to consider the reordered point to be treated as a decision variable. However, many researchers may consider that Moon and Choi [13] and Hariga and Ben-Daya [8] are independently the first authors to use the reordered point as a decision variable. This paper has been overlooked for two decades; only seven papers have quoted Horowitz and Daganzo [9] in their references. They developed a new inventory model to introduce several new variables and then used a graphical method to find the optimal order quantity. It is not only the loss of the graphical method proposed by Horowitz and Daganzo [9] but also the misfortune of the progress of inventory model. With our improvement, their inventory model will obtain the attention that deserves.

Blumenfeld et al. [2] considered inventory models with trade-off between freight expediting and safety stock costs. Horowitz and Daganzo [9] extended their model to incorporate the uncertainty of shipment size and safety stock. To the best of our knowledge, Horowitz and Daganzo [9] are the first authors to treat the reordered point as a decision variable. Since Ouyang et al. [22] constructed a new mixture inventory system to consider lost sales and backorders as measurements for shortage cost, Moon and Choi [13] and Hariga and Ben-Daya [8] independently pointed out that the reordered point should be considered as a new decision variable. Consequently, there is a trend to treat the reordered point as a decision variable. For examples, Ouyang, et al. [16], Ouyang and Chuang [18], Ouyang and Chuang [19], Ouyang and Chuang [20], Wu and Ouyang [28], Wu and Ouyang [29], Wu, et al. [27], Ouyang and Chang [15], Ouyang, et al. [17], Pan, et al. [23]. Here, we want to point out that Ouyang and Wu [21] and Chu, et al. [4], for the distribution free model, they used the reordered point as a new variable and on the other hand, for the normal distribution model, the reordered point is treated as a constant. However, we want to indicate that the first paper to take the reordered point as a decision parameter is Horowitz and Daganzo [9]. We believe, however, that their solution procedure is too complicated to be absorbed by ordinary readers and their solution is incomplete, because they do not show why their inventory model has the minimum solution. This is why this pioneer work of Horowitz and Daganzo [9] seems to have been ignored. Up to now only seven papers: Dar-El and Malmborg [6], Tyworth [24], Tyworth and Ruiz-Torres [25], Daganzo [5], Blauwens et al. [1], Namit et al. [14], and Dullaert et al. [7] have referred to Horowitz and Daganzo [9] in their references. Comparing to two other papers, Moon and Choi [13] had been cited 131 times, and Hariga and Ben-Daya [8] had been referred 130 times. Horowitz and Daganzo [9] may be overlooked by researchers which is a loss for research society. There are several related papers: Chang et al. [3], Hung et al. [10], Huang [11], Lin and Hopscotch [12] and Wang et al. [26] that will provide valuable literature review for modern development with operations research.

In this paper, we will first simplify their derivation to obtain the first derivative of the objective function. Next, we will prove that the inventory model indeed has the minimum solution. Using the same numerical examples of Horowitz and Daganzo [9], we will illustrate the saving by our improved solution procedure. We hope that this paper will provide a basis for the understanding of Horowitz and Daganzos paper and give it the attention that it deserves.

2. Notation and Assumptions

To be comparable with Horowitz and Daganzo [9], we use the same notation and assumptions as theirs.

S = safety stock,

Q = shipment size (number of parts), a decision variable,

s = reorder point,

L = lead time,

$\overline{E}_L$ = forecasted demand during the lead time,

σ_L = standard derivation of the different between predicted and actual demand (the measure of uncertainty) during the lead time,

k = safety factor: the multiple of σ_L that determines the safety stock, a decision variable,

D = demand (parts/year),

F = fixed freight cost per regular shipment,

G = fixed freight cost per shipment expedited because of stockout,

P = part value,

R = inventory cost per dollar per year,

$p(k)$ = probability of stockout during the lead time (assumed to be equal to $\Phi(-k)$, the cumulative distribution function of a unit normal random variable),

C = total annual cost.

3. Review of Previous Results of Horowitz and Daganzo [9]

Horowitz and Daganzo [9] studied the inventory model with expedited shipment when stockout occurred. We directly quote their total annual cost, $C(k,Q)$,

$$C(k,Q) = \frac{DF}{Q} + PRQ + kPR\sigma_L + \frac{DG}{Q}p(k). \tag{3.1}$$

From their assumption $p(k) = \Phi(-k) = \int_{-\infty}^{-k} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt = \int_{-\infty}^{-k} \phi(t)dt$, they knew that

$$\frac{d}{dk}p(k) = \phi(-k)\frac{d}{dk}(-k) = -\phi(k) \tag{3.2}$$

since $\phi(k)$ is the density function of the unit normal distribution with $\phi(k) = \frac{1}{\sqrt{2\pi}}e^{\frac{-k^2}{2}}$ to satisfy $\phi(-k) = \phi(k)$. They found

$$\frac{\partial}{\partial k}C(k,Q) = PR\sigma_L - \frac{DG}{Q}\phi(k), \tag{3.3}$$

and they solved for $\frac{\partial}{\partial k}C(k,Q) = 0$ which showed that

$$\phi(k^*) = \frac{1}{\sqrt{2\pi}}e^{\frac{-1}{2}(k^*)^2} = \frac{PR\sigma_L Q}{DG} \tag{3.4}$$

to yield

$$k^* = \sqrt{2\ln \frac{DG}{\sqrt{2\pi}PR\sigma_L Q}}, \tag{3.5}$$

if $\frac{DG}{\sqrt{2\pi}PR\sigma_L Q} \geq 1$ guarantee $k^* \geq 0$.

Therefore, there is a natural possible domain for the optimal solution of the order quantity to guarantee the non-negativity of k^* with Equation (3.5), as

$$\frac{DG}{\sqrt{2\pi}PR\sigma_L} \geq Q > 0. \tag{3.6}$$

For those Q satisfying $\frac{DG}{\sqrt{2\pi}PR\sigma_L Q} < 1$, Horowitz and Daganzo [9] proved that $C(k, Q)$ is an increasing function of k such that the minimum value will occur when $k = 0$. This implies that there is no safety stock. They believed that in most practical situations, the optimal value of k is positive. Hence, in their paper, they implicitly assumed the possible range for Q as

$$\frac{DG}{\sqrt{2\pi}PR\sigma_L} > Q > 0. \tag{3.7}$$

Under the restriction in Equation (3.7), they plugged Equation (3.5) into Equation (3.1) to express the total annual cost in one variable Q as

$$C(Q) = \frac{DF}{Q} + PRQ + k^* PR\sigma_L + \frac{DG}{Q}\Phi(-k^*). \tag{3.8}$$

where k^* satisfies Equation (3.5). They created a complicated procedure to solve $\frac{d}{dQ}C(Q)$ $= 0$ and then they used a graphical method to show that $\frac{d}{dQ}C(Q) = 0$ has a unique solution. They did not discuss why the inventory model has an optimal solution or whether or not the optimal solution is unique. Moreover, the graphical method may be not accurate, so researchers can only roughly estimate the optimal shipment size. We can provide an analytical method to find the optimal solution.

4. Our Improvement

Our original goal is to prove $\frac{d}{dQ}C(Q) = 0$ having a unique solution under the condition of Equation (3.6). However, we will face technical problems such that we will shrink the searching domain from Equation (3.6) to a small (In Example 1, 91%) domain then we can prove the existence and uniqueness of the optimal solution. To simplify the expression for our development, based on Equation (3.5), we assume that

$$k^*(Q) = \sqrt{2\ln \frac{DG}{\sqrt{2\pi}PR\sigma_L Q}} \tag{4.1}$$

to clearly indicate that $k(Q)$ is a function of Q. We rewrite Equation (3.8) as

$$C(Q) = \frac{DF}{Q} + PRQ + PR\sigma_L k^*(Q) + \frac{DG}{Q}\Phi(-k^*(Q)). \tag{4.2}$$

We know that $\frac{d}{dQ}\Phi(-k(Q)) = -\phi(-k(Q))\frac{dk(Q)}{dQ}$ to evaluate

$$\phi(-k^*(Q)) = \phi(k^*(Q)) = \frac{1}{\sqrt{2\pi}}\exp\Big(\frac{-(k^*(Q))^2}{2}\Big) = \frac{PR\sigma_L Q}{DG}. \tag{4.3}$$

Using Equation (4.3), it yields that

$$\frac{d}{dQ}C(Q) = \frac{-DF}{Q^2} + PR + \frac{-DG}{Q^2}\Phi(-k^*(Q)). \tag{4.4}$$

From Equation (4.4), we try to solve $\frac{d}{dQ}C(Q) = 0$, then

$$\frac{PR}{DG}Q^2 = \frac{F}{G} + \Phi(-k^*(Q)). \tag{4.5}$$

To compare with results in Horowitz and Daganzo [9], Equation (4.5) implies

$$\sqrt{\frac{DG}{PR}} = \Big(\frac{1}{Q}\sqrt{\frac{F}{G} + \Phi(-k(Q))}\Big)^{-1}. \tag{4.6}$$

If we multiple $\sigma_L\frac{PR}{DG}$ on both sides of Equation (4.6), then it derives the same results as Equation (24) of Horowitz and Daganzo [9]. However, our derivation is clear. Their procedure contains too many new expressions which is misleading. Moreover, in Horowitz and Daganzo [9], they did not try to prove the existence and uniqueness for the solution of Q with respect to Equation (4.5). Instead, they defined a new expression of $x = \frac{DG}{PR\sigma_L Q}$. The condition of $\frac{DG}{\sqrt{2\pi}PR\sigma_L} > Q$ is equivalent to $x > \sqrt{2\pi}$. From observation of the graph, they claimed that if the values of x are between $\sqrt{2\pi}$ and 5.2 that will imply an extremely high probability of stock out and degree of uncertainty, which could violate the basic assumptions of the model. Hence, in their paper, they implicitly assumed that $\infty > x > 5.2$ that means

$$\frac{5DG}{26PR\sigma_L} > Q > 0. \tag{4.7}$$

They used a graphical method to locate the root of Q for Equation (4.5).

Now, we begin to discuss why Equation (4.5) has a unique solution under the condition of Equation (3.6) as $\frac{DG}{\sqrt{2\pi}PR\sigma_L} \geq Q > 0$. However, we will slightly shrink the domain to $Q_U \geq Q \geq Q_L$ such that we can analytical prove the existence and uniqueness of the optimal solution with $\dfrac{DG}{\sqrt{2\pi}PR\sigma_L} > Q_U$ and $Q_L > 0$ that will be explained in the following.

Motivated by Equation (4.5), we assume an auxiliary function as

$$g(Q) = \frac{PR}{DG}Q^2 - \frac{F}{G} - \Phi(-k(Q)), \tag{4.8}$$

with $\dfrac{d}{dQ}C(Q) = \dfrac{DG}{Q^2}g(Q)$.

Our original goal is to find a lower bound Q_L and an upper bound Q_U and to prove that

$$\text{(a)}\ \ g(Q_L) < 0, \tag{4.9}$$

$$\text{(b)}\ \ g(Q_U) > 0, \quad \text{and} \tag{4.10}$$

$$\text{(c)}\ \ g'(Q) > 0 \ \text{ for }\ Q_U > Q > Q_L. \tag{4.11}$$

Therefore, if our selection of Q_L and Q_U can satisfy the above mentioned conditions of Equations (4.9)−(4.11), then by the Intermediated Value Theorem, we can prove that under

$$Q_U \geq Q \geq Q_L. \tag{4.12}$$

then $g(Q) = 0$ has a unique solution.

Here, we provide an explanation why we need $Q_L > 0$. The reason for $\dfrac{DG}{\sqrt{2\pi}PR\sigma_L} > Q_U$ will be self-explanation after our findings of Table 1.

If we try to compute $g(0)$ then we need to derive $\Phi(-k(0))$ such that we have to find $k(0)$. From our definition of Equation (4.1), we can not compute $k(0)$, even we know that

$$\lim_{Q\to 0^+} k(Q) = \lim_{Q\to 0^+} \sqrt{2\ln \frac{DG}{\sqrt{2\pi}PR\sigma_L Q}} = \infty \tag{4.13}$$

and then $\lim\limits_{Q\to 0^+} \Phi(-k(Q)) = \Phi(-\infty) = 0$, to imply that $\lim\limits_{Q\to 0^+} g(Q) = \dfrac{-F}{G} < 0$. However, to extend the domain of $g(Q)$ from $Q > 0$ to $Q \geq 0$ with the assumption

$$g(0) = \lim_{Q\to 0^+} g(Q) \tag{4.14}$$

that is too technical in inventory system, so we will pick a point very close to zero, say Q_L satisfying the wanted property $g(Q_L) < 0$.

We obtain that

$$\frac{d}{dQ}g(Q) = \frac{2PR}{DG} = \frac{\dfrac{1}{Q}\exp\left(\dfrac{-1}{2}(k^*(Q))^2\right)}{\sqrt{4\pi \ln \dfrac{DG}{2\pi PR\sigma_L Q}}}. \tag{4.15}$$

To prove $\frac{d}{dQ}g(Q) > 0$ for $Q_U > Q > Q_L$ is equivalent to verify that

$$\ln\Big(\frac{DG}{\sqrt{2\pi}PR\sigma_L Q}\Big) > \frac{\sigma_L^2}{8Q^2} \tag{4.16}$$

for $Q_U > Q > Q_L$, since $\exp\Big(\frac{(k^*(Q))^2}{2}\Big) = \frac{DG}{\sqrt{2\pi}PR\sigma_L Q}$.

Motivated by Equation (4.16), we assume the second auxiliary function

$$f(Q) = \ln\Big(\frac{DG}{\sqrt{2\pi}PR\sigma_L Q}\Big) - \frac{\sigma_L^2}{8Q^2} \tag{4.17}$$

such that to verify $\frac{d}{dQ}g(Q) > 0$ is equivalent to prove that $f(Q) > 0$. Hence, we convert the condition of Equation (4.11) to

$$f(Q) > 0 \quad \text{for } Q_U > Q > Q_L. \tag{4.18}$$

We will try to find a feasible domain, $Q_U \geq Q \geq Q_L$ that satisfies $f(Q) > 0$, for $Q_U > Q > Q_L$.

It then follows that

$$\frac{d}{dQ}f(Q) = \frac{1}{Q^3}\Big(\frac{\sigma_L}{2} - Q\Big)\Big(\frac{\sigma_L}{2} + Q\Big) < 0, \tag{4.19}$$

to imply that $f(Q)$ is a decreasing function of Q for $Q \geq \frac{\sigma_L}{2}$. Hence, to prove $f(Q) > 0$ for $Q_U > Q > Q_L$ of Equation (4.18) is equivalent to select Q_U with

$$f(Q_U) > 0. \tag{4.20}$$

On the other hand, $\frac{Q_L}{2}$ is a possible candidate for Q_L.

We begin to show that $g\Big(\frac{\sigma_L}{2}\Big) < 0$ with the data from Horowitz and Daganzo [9]. From the numerical example of Horowitz and Daganzo [9], we know that $D = 52000$, $F = 2500$, $G = 2500$, $P = 100$ and $R = 0.2$. In Figures 2, 3 and 4 of Horowitz and Daganzo [9], they sketched five curves where $\frac{F}{G}$ equals to 0.2, 0.5, 1, 2 and 5, such that we assume that $0.2 \leq \frac{F}{G} \leq 5$.

There are two values of σ_L : 400 and 100, so we execute two examinations. When $\sigma_L = 400$ then $\frac{PR}{4DG}\sigma_L^2 \approx 6.2 \times 10^{-3}$. We compute

$$\frac{PR}{DG}Q^2 - \frac{F}{G} = -0.9938 < 0 \tag{4.21}$$

when we take $Q = \dfrac{\sigma_L}{2}$.

When $\sigma_L = 100$ then $\dfrac{PR}{4DG}\sigma_L^2 \approx 3.8 \times 10^{-4}$. We compute

$$\frac{PR}{DG}Q^2 - \frac{F}{G} = -0.99962 < 0 \tag{4.22}$$

when we take $Q = \dfrac{\sigma_L}{2}$.

Consequently, we assume that

$$Q_L = \frac{\sigma_L}{2} \tag{4.23}$$

to imply that

$$g\Big(\frac{\sigma_L}{2}\Big) = \frac{PR}{4DG}\sigma_L^2 - \frac{F}{G} - \Phi\Big(-k\Big(\frac{\sigma_L}{2}\Big)\Big) < \frac{PR}{4DG}\sigma_L^2 - \frac{F}{G} < 0 \tag{4.24}$$

then the condition of Equation (4.9) as $g\Big(\dfrac{\sigma_L}{2}\Big) < 0$ is satisfied.

Next, we begin to explain how to select our desired Q_U that must satisfy two required properties of Equations (4.20) and (4.10):

(a) $f(Q_U) > 0$, and

(b) $g(Q_U) > 0$.

We define a new expression

$$Q_{n,m} = \Big(\frac{n}{m}\Big)\frac{DG}{\sqrt{2\pi}PR\sigma_L}, \tag{4.25}$$

where n and m are positive integer with $n < m$. We compute $f(Q_{n,m})$ to derive that

$$f(Q_{n,m}) = \Big(\ln\frac{n}{m}\Big) - \frac{\pi}{4}\Big(\frac{mPR\sigma_L^2}{nDG}\Big)^2. \tag{4.26}$$

For $m = 100$ and $91 \le n \le 100$, we list the computation results in the next table.

Table 1: When $m = 100$, and $91 \le n \le n$, values of $f(Q_{n,m})$.

n	91	92	93	94	95
$f(Q_{n,100})$	0.093736	0.082819	0.72020	0.061337	0.050766
n	96	97	98	99	100
$f(Q_{n,100})$	0.040306	0.029953	0.019707	0.009565	-0.000476

Next, we consider our requirement of $g(Q_U) > 0$ such that we will compute

$$g(Q_{n,m}) = \frac{DG}{2\pi PR}\Big(\frac{n}{m\sigma_L}\Big)^2 - \frac{F}{G} - \Phi\Big(-\sqrt{2\ln\frac{m}{n}}\Big). \tag{4.27}$$

However, we have a technical problem to evaluate $\Phi\Big(-\sqrt{2\ln\frac{m}{n}}\Big)$. On the other hand, we know that

$$\Phi\Big(-\sqrt{2\ln\frac{m}{n}}\Big)=\int_{-\infty}^{-\sqrt{2\ln(m/n)}}\frac{1}{\sqrt{2\pi}}e^{\frac{-1}{t^2}}dt<\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}}e^{\frac{-1}{t^2}}dt=\frac{1}{2}. \tag{4.28}$$

Therefore, we assume the third auxiliary function

$$h(Q_{n,m})=g(Q_{n,m})+\Phi\Big(-\sqrt{2\ln\frac{m}{n}}\Big)-\frac{1}{2} \tag{4.29}$$

for $m=100$ and $91\le n\le 100$, and then we list the computation results for $h(Q_{n,m})$ in the following table.

Table 2: When $m=100$, and $91\le n\le n$, values of $f(Q_{n,100})$.

n	91	92	93	94	95
$h(Q_{n,100})$	3.854	3.973	4.092	4.213	4.335
n	96	97	98	99	100
$h(Q_{n,100})$	4.459	4.584	4.710	4.837	4.966

Based on Table 1 and Table 2, if we select

$$Q_U=\Big(\frac{23}{25}\Big)\frac{DG}{\sqrt{2\pi}PR\sigma_L} \tag{4.30}$$

to yield two desired properties:

$$g(Q_U)>0, \tag{4.31}$$

and

$$f(Q_U)>0. \tag{4.32}$$

Consequently, we obtain that $f(Q)>0$ for $Q_U\ge Q\ge Q_L$ with its minimum value $f(Q_U)>0$ and then $g'(Q)>0$ for $Q_U>Q>Q_L$ which means Equation (4.11) is satisfied.

Consequently, $g'(Q)>0$ for $Q_U>Q>Q_L$, so $g(Q)$ is an increasing function from $g(Q_L)<0$ to $g(Q_U)>0$. Therefore, there is a unique solution for $g(Q)=0$, say Q^*. Finally, we will try to prove that Q^* is the minimum point.

From $g(Q)<0$ for $Q_L<Q<Q^*$, and $g(Q)>0$ for $Q^*<Q<Q_U$, using the relation $\frac{d}{dQ}C(Q)=\frac{DG}{Q^2}g(Q)$, it yields that $\frac{d}{dQ}C(Q)<0$ for $Q_L<Q<Q^*$, and $\frac{d}{dQ}C(Q)>0$ for $Q^*<Q\le Q_U$. We know that $C(Q)$ decreases for $Q_L\le Q\le Q^*$ and $C(Q)$ increases

for $Q^* < Q \leq Q_U$ so Q^* is the minimum point. Consequently, under the restriction of Equation (4.12), we show that the inventory model has a unique optimal solution.

5. Numerical Examples

To be compatible with Horowitz and Daganzo [9], we consider the same numerical examples with the following data: $D = 52000$ parts per year, $F = \$2500$ per shipment, $G = \$2500$ per shipment, $P = \$100$ per part, $R = 0.2$ per year (20%), $L = 7$ days, $\sigma_L = 400$ parts for the first example, and $\sigma_L = 100$ parts for the second example.

Example 1. We find that $g(2663) = -1.008 \times 10^{-4}$ and $g(2664) = 6.726 \times 10^{-4}$, which is consistent with our results of $g(Q)$ as an increasing function of Q. We know the optimal real solution, say Q^*, satisfying $2663 < Q^* < 2664$. However, this research is conducted for automobile parts so we need to seek the integer solution. From $C(2663) = 117196.457$ and $C(2664) = 117196.463$. Therefore, the optimal order quantity is $Q^* = 2663$.

Comparing with Horowitz and Daganzo [9], they derived that $Q^* = 2700$ and $C^* = 117210$, our saving is 13.54.

Comparing our analytic approach with the graphic method of Horowitz and Daganzo [9], we can accurately obtain the optimal integer solution and the improvement is $\dfrac{2700 - 2663}{2663} = 1.39\%$.

Example 2. We find that $g(2569) = -4.172 \times 10^{-4}$ and $g(2570) = 3.662 \times 10^{-4}$, which is consistent with our results of $g(Q)$ as an increasing function of Q. We know the optimal real solution, say Q^*, satisfying $2569 < Q^* < 2570$. As in Example 1, we search for the optimal positive integer solution. From $C(2569) = 107081.7471$ and $C(2570) = 107081.7466$. Hence the optimal order quantity is $Q^* = 2570$.

Comparing with Horowitz and Daganzo [9], they derived that $Q^* = 2600$ and $C^* = 107130$, so our saving is 48.25.

Once again, our analytical approach derives the optimal solution and the improvement is $\dfrac{2600 - 2570}{2570} = 1.17\%$.

From the two examples from Horowitz and Daganzo [9], we demonstrate that our approach can accurately find the optimal point. Our improvement for saving cost may look like insignificant. On the other hand, for examples 1and 2, we reduce the production of 37 and 30 cars respectively and also save money that is a meaningful action to against the global warming.

6. Further Discussion on the Possible Range for Minimum Solution

Here, we try to offer a further discussion of our restriction of the range of Q. From the theoretical result of Horowitz and Daganzo [9], they will use graphical method to locate an approximated optimal solution for $\dfrac{DG}{PR\sigma_L} > Q > 0$ so that the search range is $25931.26 > Q > 0$.

Based on observation, they reduced the search range to $\dfrac{5DG}{26PR\sigma_L} > Q > 0$ that is $12500 > Q > 0$. However, they did not insure that in that range the inventory model has an optimal solution. Our possible range for Q is $\dfrac{23DG}{25PR\sqrt{2\pi}\sigma_L} \geq Q \geq \dfrac{\sigma_L}{2}$, then our search range is $23856.76 > Q > 50$.

We compare the coverage of Horowitz and Daganzo [9] and ours to compute that

$$\frac{12500-0}{25931.26-0} = 48.20\%, \tag{5.1}$$

and

$$\frac{23856.76-50}{25931.26-0} = 91.81\%, \tag{5.2}$$

to reveal that they only cover 48% of the possible domain of Q to guarantee the non-negativity of $k^*(Q)$ and they did not discuss the existence and uniqueness of the optimal order quantity.

On the other hand, we cover more than 91% of the possible domain of Q. More importantly, we can prove the existence and uniqueness of the optimal order quantity. We can always prove that $\dfrac{23DG}{25PR\sqrt{2\pi}\sigma_L} > \dfrac{5DG}{26PR\sigma_L}$ such that our upper bound for the possible range is better than that of Horowitz and Daganzo [9]. On the other hand, our lower bound of $\dfrac{\sigma_L}{2}$ is always worse than that of Horowitz and Daganzo [9], which means that we have a worse lower bound but a much better upper bound. Moreover, in our possible range, we can insure the inventory model has an optimal solution. We sacrifice a little range of $\left(0, \dfrac{\sigma_L}{2}\right)$ to guarantee the existence and uniqueness of the optimal solution. From the optimal solution for the second example, $Q^* = 2570$ with $\dfrac{\sigma_L}{2} = 50$ may indicate that Q^* is far away from $\dfrac{\sigma_L}{2}$ such that our restriction of $Q \geq \dfrac{\sigma_L}{2}$ can be accepted as a reasonable condition.

7. Conclusion

We provide a detailed analytical procedure to show that after shrinking the original domain (up to 91%) we can prove the existence and uniqueness of the optimal solution. In Horowitz and Daganzo (1986), they applied a graphical method to locate an approximated optimal solution where their searching sub-domain is about 48% of the original domain. The managerial meaning of our findings is that we produce fewer cars and also save more money that is beneficial both for environment and economy. Our derivations can help researchers develop their mathematical approach to locate the optimal solution for future inventory models.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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