International Journal of
Information
and
Management
Sciences

International Journal of Information and Management Sciences
29 (2018), 57-88. DOI:10.6186/IJIMS.2018.29.1.3

Carbon-Constrained Deteriorating Inventory Model When Inventory Stimulates Demand

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Abstract

Nowadays, environmental protection issues have received more and more attention due to climate change and global warming. Many countries around the world have implemented carbon emission regulations to curb the greenhouse gas emissions to slow down the process and prevent it from deteriorating ecological system, In this paper we develop a carbon-constrained deteriorating inventory model when inventory stimulates demand. We first characterize the profit maximizing inventory replenishment strategy and investigate the impacts of preservation technology investment and carbon emission parameters on retailer's inventory replenishment strategy under the Carbon Cap-and-Trade policy. We also make some extensions to the model with the Carbon Offset policy. Finally, a numerical example and sensitivity analysis are presented to illustrate the theoretical results and obtain some managerial insights, which is followed by concluding remarks and future research.

Keywords: Inventory, Stock-dependent demand, Deterioration, Preservation technology investment, Carbon emissions.

1. Introduction

Human-induced climate change and global warming have posed enormous risks to our environment, economies and societies, they have become areas of growing concern for scientists, environmentalists and public policymakers over past two decades. In order to reduce greenhouse gases and thwart the threat of climate change and global warming, the Kyoto Protocol is the first international agreement bound by law under the UN climate convention. The purpose of the Kyoto Protocol is to reduce greenhouse gas emissions from industrial countries to 5.2% below 1990 levels between 2008 and 2012. Emissions Trading (also known as cap and trade) is the only administration-based mechanism of the three Kyoto Protocol mechanisms. The Carbon Cap-and-Trade system has been designed to reduce carbon emission by utilizing market mechanisms. The system puts a mandatory limit on emissions to all the firms, and allows firms to buy and sell rights to emit carbon dioxide within the cap. Many countries or regions have made either voluntary or regulatory efforts to reduce their carbon emissions to meet targets for the carbon emissions reduction set by the Kyoto Protocol. In response to the regulations on carbon emissions reduction, the impact of carbon emissions on operations management has drawn more academic attention. Recently, [6] first showed how carbon emission parameters can be added to various inventory models. Based on the classic EOQ model, [2] and [21] then developed environmental inventory models, and the authors also explored the different impacts of carbon emission parameters on the inventory replenishment decision. Meanwhile, [22] extended the same issue to the price-sensitive demand with the Carbon Cap-and-Trade policy. [32] proposed a classical single-period newsvendor model with various carbon emission regulatory policies. [7] addressed the inventory management problems by incorporating sustainability considerations. [9] conducted a research to analyze the impact of carbon emission regulatory policies on the classical EOQ model. On the basis of their study, they concluded that operational adjustments alone could indeed be effective in reducing emissions. Since transportation is a main source of carbon emission, [40] analyzed the operations and loion decisions for a manufacturer under the Carbon Cap-and-Trade policy. [42] studied the multi-item newsvendor problem to the multi-item production planning problem with finite capacity and the Carbon Cap-and-Trade policy. They also derived the optimal policy of production and carbon trading decisions. [5] explored the integration of factors affecting the environmental impact of transportation and inventory within the traditional EOQ model.

However, all these researches mentioned above fail to take into account the empirical demand stimulating role of retail inventories. In contrast to constant or stochastic demand rates, for certain items, the displayed stock level has a positive impact on sales and profits. This happens because large amounts of inventory are more visible than small ones, and this increased visibility stimulates demand by creating cross-selling and impulse-buy opportunities. Large amounts of inventory might signal a popular product, or provide consumers with an assurance of high service levels. Marketing researchers and practitioners have recognized the phenomenon that high inventories might stimulate demand in years. [38] first observed that displayed inventory can help induce greater sales. [25] noted that the presence of inventory has a motivational effect on the people around it, and large piles of goods displayed in a supermarket can lead the customers to buy more. [11] also noted that impulsive buying categories have higher space elasticities, which is consistent with the interpretation that space has a causal effect on sales and not the converse. Hence, higher inventories not only improve service level to gain the competitive advantages in business, but stimulate demand by serving as a promotional tool. Due to the facts, a number of authors have developed the EOQ models that focused on stock-dependent demand rate patterns. [18] assumed that the demand rate was a function of initial stock level. [4, 3] considered a power-form inventory-level-dependent demand rate, which would decline along with the stock level throughout the entire cycle. [10] considered a situation in which the stock-dependent demand rate was down to a given level of inventory, beyond which it is a constant. [17], [1] and [28] relaxed the assumption of a constant holding cost in [3]. Later, [37] extended Datta and Pal's [10] model to allow shortages, where the unsatisfied demand is backlogged at a fixed fraction of the constant demand rate.

Furthermore, deterioration is a common phenomenon in daily life because of poor storage and preservation quality. [26] first considered the inventory model for perishable items with a linearly stock-dependent demand. [27] then analyzed an EOQ model for perishable items when inventory stimulates demand, allowing shortages and shortage-dependent partial backlogging. Meanwhile, [16] and [15] extended the models of [10] and [17] by considering deteriorating items. Lately, based on the result of [27], [12] provided a deteriorating inventory model with stock-dependent demand and reciprocal time-dependent backlogging rate. [35] then established an economic production quantity model for deteriorating items and studied pricing and inventory policies where demand depends on both price and inventory level. More recently, [23] and [41] incorporated effects of inflation, deterioration and stock-dependent demand rates to develop an inventory/production model over a finite planning horizon. [33] and [29] proposed deterministic inventory models with stock-dependent demand under trade credit policy. [39] and [8] further extended Dye and Ouyang's model [12] to the case of non-instantaneous deteriorating items because most goods would have a span of maintaining quality or original condition. [30] then extended this issue to the varying rate of deterioration and stock-dependent demand. [34] similarly proposed an inventory model for non-instantaneous deteriorating items with demand influenced by both displayed stock level and selling price under delay in payment.

However, the deterioration rate of goods in the above mentioned papers is viewed as an exogenous variable, which is not subject to control. Although the deterioration of goods is a natural process that cannot be stopped, it can be slowed down with the corrective actions taken when handling and advancing preservation equipments. To express agreement with the practical inventory situation, [20] first incorporated the effect of preservation technology investment into the classical deteriorating inventory model. Their model aims to determine simultaneously both the optimal replenishment and preservation technology investment strategies. [24] relaxed the assumption of a constant demand rate in the model of [20] to study the inventory replenishment and preservation technology investment decisions when inventory stimulates demand. Lately, [13] extended the model of [20] to a generalized deteriorating inventory system, and further showed in a rigorous way that a higher preservation technology investment indeed leads to a higher service rate. [19] extended this issue to the cases of price-sensitive demand. [14] and [31] adopted a time-dependent demand function and preservation technology investment to model the finite time horizon inventory system with trade credit financing. More recently, [36] extended the model of [13] to consider a joint location and preservation technology investment decision-making problem for non-instantaneous deteriorating items under trade credit.

The inventory system for deteriorating items with stock-dependent demand has been a topic of study for a long time, but little is known about the effects of preservation technology investment and carbon regulatory policy. Since carbon is the basic element in fossil energy, cutting carbon emissions equals to cost savings and operational efficiency. In this paper, we incorporate environmental protection considerations into inventory replenishment decision making when inventory level stimulates demand. In addition, we use general forms for the time-dependent backlogging rate and productivity of preservation technology investment. The main emphasis of this paper is to determine the optimal replenishment and preservation technology investment strategies which maximize the total profit per unit time under different carbon emission regulatory policies. We first begin with the model formulation for the inventory system with the Carbon Cap-and-Trade policy. Using the obtained theoretical results, we then extend the model to the case with the Carbon Offset policy. Furthermore, we also investigate the impacts of preservation technology investment and carbon cost parameters on retailer's inventory replenishment strategy. A numerical example and sensitivity analysis are presented to illustrate the theoretical results and obtain some managerial insights, which is followed by concluding remarks and future research. To keep the presentation simple, all proofs are put in the appendix at the end of this paper.

2. Notation and Assumptions

2.1. Notation

To develop the mathematical model of inventory system, the notation adopted in this paper is as below:

$A =$ the replenishment cost per order.

$C =$ the purchase cost per unit.

$S =$ the selling price per unit, where $S > C$.

$C_1 =$ the holding cost per unit per unit time.

$C_2 =$ the backorder cost per unit per unit time.

$C_3 =$ the opportunity cost (i.e., goodwill cost) per unit.

$\widehat{A} =$ the amount of fixed carbon emissions per order.

$\widehat{C} =$ the amount of carbon emissions associated per unit purchased.

$\widehat{C}_1 =$ the amount of carbon emissions per unit of inventory held per unit time.

$\varpi =$ the carbon cap.

$w =$ the maximum capital constraint.

$t_1 =$ the time at which the inventory level reaches zero.

$T =$ the length of the inventory cycle.

$\theta =$ the deterioration rate, a fraction of the on-hand inventory.

$\xi =$ the preservation technology investment per unit time for reducing deterioration rate in order to preserve the products, where $0 \leq \xi \leq w$.

$E =$ the carbon price per unit emission of carbon.

$I(t) =$ the level of inventory at time t.

$TP(t_1, T, \xi) =$ the total profit per inventory cycle without considering carbon emission costs.

$CE(t_1, T, \xi) =$ the amount of carbon emissions per inventory cycle.

$\Pi(t_1, T, \xi) =$ the total profit per unit time.

2.2. Assumptions

In addition, the following assumptions are imposed:

1. The replenishment rate is infinite and lead time is zero.
2. The time horizon of the inventory system is infinite.
3. The demand rate function $D(t)$ is deterministic and a function of instantaneous stock level $I(t)$. When inventory is positive, $D(t)$ is given by:
$$D(t) = \alpha + \beta I(t),\ 0 \leq t \leq t_1,$$
and when inventory is negative, $D(t)$ is given by:
$$D(t) = \alpha,\ t_1 < t \leq T,$$
where $\alpha > 0$ and $0 < \beta < 1$ are known as scale and shape parameters respectively.
4. There is no repair or replacement of deteriorated units. The items will be withdrawn from warehouse immediately as they become deteriorated.
5. The reduced deterioration rate, $m(\xi)$, is an increasing function of the preservation technology investment ξ, where $\lim_{\xi\to\infty} m(\xi) = \theta$.
6. Shortages are allowed. The fraction of shortages backordered is a decreasing function $b(x)$, where x is the waiting time up to the next replenishment, and $0 \leq b(x) \leq 1$ with $b(0) = 1$. Note that if $b(x) = 1$ (or 0) for all x, then shortages are completely backlogged (or lost).

3. The Model

Utilizing the above notation and assumptions, the depletion of the inventory occurs due to the combined effects of the demand and deterioration in the interval $(0, t_1)$ and the demand backlogged in the interval (t_1, T), respectively. Hence, the variation of inventory level, $I(t)$, with respect to time can be described by the following differential equation:

$$\frac{dI(t)}{dt} = \begin{cases} -\alpha - \beta I(t) - [\theta - m(\xi)]\, I(t), 0 < t < t_1, \\ \\ -\alpha b(T-t), \qquad\qquad\qquad t_1 < t < T, \end{cases} \tag{3.1}$$

with boundary condition $I(t_1) = 0$. The solution of (3.1) is

$$I(t) = \begin{cases} \dfrac{\alpha\left\{e^{[\beta+\theta-m(\xi)](t_1-t)} - 1\right\}}{\beta + \theta - m(\xi)}, 0 < t < t_1, \\ \\ -\alpha \int_{t_1}^{t} b(T-u)du, \qquad\qquad t_1 < t < T. \end{cases} \tag{3.2}$$

Hence, the total profit and the amount of carbon emissions per inventory cycle can be respectively calculated as follows:

$$
\begin{aligned}
TP(t_1,T,\xi) &= \left\{\begin{array}{l}\text{sales revenue} - \text{ordering cost} - \text{purchase cost} - \text{holding cost} \\ -\text{shortage cost} - \text{opportunity cost} - \text{preservation technology cost}\end{array}\right\} \\
&= S\left\{\int_0^{t_1} \alpha + \beta I(t)dt + \alpha \int_{t_1}^{T} b(T-t)dt\right\} - A \\
&\quad -C\left[I(0) + \alpha\int_{t_1}^{T} b(T-t)dt\right] - C_1\int_0^{t_1} I(t)dt \\
&\quad -\alpha C_2\int_{t_1}^{T} (T-t)\, b(T-t)dt - \alpha C_3 \int_{t_1}^{T} 1-b(T-t)dt - \xi T \\
&= \frac{\alpha\left[\beta S - C_1 - C\left(\beta+\theta-m(\xi)\right)\right]}{\left[\beta+\theta-m(\xi)\right]^2}\left\{e^{[\beta+\theta-m(\xi)]t_1} - \left[\beta+\theta-m(\xi)\right]t_1 - 1\right\} \\
&\quad +\alpha\left(S-C+C_3\right)t_1 + \alpha\int_{t_1}^{T}\left[S-C+C_3-C_2\left(T-t\right)\right]b(T-t)dt \\
&\quad -A - \xi T - \alpha C_3 T
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
CE(t_1,T,\xi) = \widehat{A} &+ \frac{\alpha\widehat{C}_1\left\{e^{[\beta+\theta-m(\xi)]t_1} - \left[\beta+\theta-m(\xi)\right]t_1 - 1\right\}}{\left[\beta+\theta-m(\xi)\right]^2} \\
&+\alpha\widehat{C}\left\{\frac{e^{[\beta+\theta-m(\xi)]t_1}-1}{\beta+\theta-m(\xi)} + \int_{t_1}^{T} b(T-t)dt\right\}.
\end{aligned} \tag{3.4}
$$

In this paper we begin with the model formulation for the inventory system with the Carbon Cap-and-Trade policy. Under the Carbon Cap-and-Trade policy, the retailer is given an initial cap for the carbon emissions and allowed to buy and sell rights to emit within the cap. If the retailer's amount of carbon emissions exceeds its carbon cap ϖ, it has to buy allowances from the carbon trading market. However, if the retailer's amount of carbon emissions is lower than its carbon cap, it can sell the allowances to generate revenue. The total profit per unit is therefore given by

$$
\begin{aligned}
\Pi(t_1,T,\xi) &= \frac{TP(t_1,T,\xi)}{T} - E\left[\frac{CE(t_1,T,\xi)}{T} - \varpi\right] \\
&= \frac{1}{T}\Bigg\{\alpha\left\{\beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\} \\
&\quad \times\frac{e^{[\beta+\theta-m(\xi)]t_1} - \left[\beta+\theta-m(\xi)\right]t_1 - 1}{\left[\beta+\theta-m(\xi)\right]^2} + \alpha\left[S - (C+E\widehat{C}) + C_3\right]t_1 \\
&\quad +\alpha\int_{t_1}^{T}\left[S - (C+E\widehat{C}) + C_3 - C_2\left(T-t\right)\right]b(T-t)dt \\
&\quad -\left(A + E\widehat{A}\right)\Bigg\} - \xi - \left(\alpha C_3 - E\varpi\right).
\end{aligned} \tag{3.5}
$$

The problem now can be formulated as choosing the replenishment and preservation technology strategies in order to solve the following nonlinear constrained optimization problem:

$$\max_{t_1,T,\xi} \quad \Pi(t_1,T,\xi), \quad \text{s.t. } 0 < t_1 < T \quad \text{and} \quad 0 \leq \xi \leq w.$$

Let $f(t_1,T,\xi) \equiv TP(t_1,T,\xi) - E\left[CE(t_1,T,\xi) - \varpi T\right]$, if $0 < t_1^* < T^*$ and $0 < \xi^* < w$, the necessary condition for maximizing $\Pi(t_1,T,\xi)$ are

$$\begin{aligned}
\frac{\partial \Pi(t_1,T,\xi)}{\partial t_1} &= \frac{1}{T}\frac{\partial f(t_1,T,\xi)}{\partial t_1} \\
&= \frac{\alpha}{T}\left\{\beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C})\left[\beta + \theta - m(\xi)\right]\right\}\frac{e^{[\beta+\theta-m(\xi)]t_1} - 1}{\beta + \theta - m(\xi)} \\
&\quad + \frac{\alpha}{T}\left[S - (C + E\widehat{C}) + C_3\right]\left[1 - b\left(T - t_1\right)\right] + \frac{\alpha C_2}{T}(T - t_1)b\left(T - t_1\right) \\
&= 0
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\frac{\partial \Pi(t_1,T,\xi)}{\partial T} &= \frac{1}{T}\frac{\partial f(t_1,T,\xi)}{\partial T} - \frac{f\left(t_1,T,\xi\right)}{T^2} \\
&= \frac{1}{T}\left\{\alpha\left[S - (C + E\widehat{C}) + C_3 - C_2\left(T - t_1\right)\right]b(T - t_1) - \xi - (\alpha C_3 - E\varpi)\right\} \\
&\quad - \frac{f(t_1,T,\xi)}{T^2} \\
&= 0
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\frac{\partial \Pi(t_1,T,\xi)}{\partial \xi} &= \frac{1}{T}\frac{\partial f(t_1,T,\xi)}{\partial \xi} \\
&= -\frac{\alpha m'(\xi)}{T}\Bigg\{\left\{\beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C})\left[\beta + \theta - m(\xi)\right]\right\} \\
&\quad \times \frac{e^{[\beta+\theta-m(\xi)]t_1}\left[\beta + \theta - m(\xi)\right]t_1 + \left[\beta + \theta - m(\xi)\right]t_1 - 2e^{[\beta+\theta-m(\xi)]t_1} + 2}{\left[\beta + \theta - m(\xi)\right]^3} \\
&\quad - (C + E\widehat{C})\frac{e^{[\beta+\theta-m(\xi)]t_1} - \left[\beta + \theta - m(\xi)\right]t_1 - 1}{\left[\beta + \theta - m(\xi)\right]^2}\Bigg\} - 1.
\end{aligned} \tag{3.8}$$

Eqs. (3.6) and (3.7), after some rearrangements, can be then rewritten as

$$\begin{aligned}
0 &= \left\{\beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C})\left[\beta + \theta - m(\xi)\right]\right\}\frac{e^{[\beta+\theta-m(\xi)]t_1} - 1}{\beta + \theta - m(\xi)} \\
&\quad + S - (C + E\widehat{C}) + C_3 - [S - (C + E\widehat{C}) + C_3 - C_2(T - t_1)]b(T - t_1)
\end{aligned} \tag{3.9}$$

and

$$\Pi(t_1,T,\xi) = \alpha[S - (C + E\widehat{C}) + C_3 - C_2\left(T - t_1\right)]b(T - t_1) - \xi - \left(\alpha C_3 - E\varpi\right). \tag{3.10}$$

Here we observe from (3.5) that if the retailer closes the inventory system (i.e., $t_1 = 0$, $\xi = 0$ and $T \to \infty$), the cost of losing all sales per unit time is the total goodwill cost, which is equal to αC_3. Since the retailer can sell its carbon credits to the carbon trading market under the Carbon Cap-and-Trade policy, the earning from carbon trading market is $E\varpi$. If $E\varpi \geq \alpha C_3$, it implies that losing sales all the time is more beneficial than operating an inventory system. It is not realistically feasible. In addition, from (3.5), we can also find that $\lim_{\xi\to\infty}\Pi(t_1, T, \xi) = -\infty < 0$. Because the inventory system should not be operated if $\Pi(t_1, T, \xi) < 0$, the value of $f(t_1, T, \xi)$ must be non-negative. Finally, as pointed out by [24] that $\beta[S-(C+E\widehat{C})]$ is the benefit received from a unit of inventory and $(C_1 + E\widehat{C}_1) + (C + E\widehat{C})[\theta - m(\xi)]$ is the cost due to a unit of inventory. If $\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\,[\beta+\theta-m(w)] \geq 0$, it implies that the sales revenue received from every unit inventory is greater than or equal to the cost of every item. Under this circumstance, building inventory is profitable and we should display inventory to the maximum as possible as we can implying we can not find a finite solution such that $\Pi(t_1, T, \xi)$ is maximum. In the following sections, we restrict ourselves to the case in which the retailer has finite positive total profit per unit time. More specifically, without loss of generality, we make the following assumptions:

Assumption 1. The cost of losing all sales per unit time is greater than the revenue of carbon cap per unit time , i.e., $E\varpi < \alpha C_3$.

Assumption 2. For a given feasible ξ, $\mathbf{\Omega} = \{(t_1, T) : 0 < t_1 < T, f(t_1, T, \xi) \geq 0\}$ is a nonempty set.

Assumption 3. The benefit received from every unit inventory is less than the cost of every item, i.e., $\beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C})\,[\beta + \theta - m(w)] < 0$.

Considering Eqs. (3.6)–(3.8), one can notice that solving the problem analytically is not possible because of the presence of nonlinearity, we solve the maximization problem stepwise. First, we find the optimal replenishment policy for a given preservation technology investment. Then using the optimal replenishment strategy for every preservation technology investment ξ, we can find the profit maximizing preservation technology investment in the operating region $0 \leq \xi \leq w$.

From (3.7), it is straightforward to see that $\frac{\partial\Pi(t_1,T,\xi)}{\partial T} < 0$ on $\mathbf{\Omega}$ when $S-(C+E\widehat{C})+C_3 - C_2\,(T - t_1) \leq 0$. The following lemma shows that the optimal solution must lie on the feasible side of the constraint boundary if $S - (C + E\widehat{C}) + C_3 - C_2\,(T - t_1) \leq 0$. For the convenience of discussion, we divide $\mathbf{\Omega}$ into two parts, $\mathbf{\Omega}_1 := \mathbf{\Omega} \cap \{(t_1, T) : S - (C + E\widehat{C}) + C_3 - C_2(T - t_1) \leq 0\}$ and $\mathbf{\Omega}_2 := \mathbf{\Omega} \setminus \mathbf{\Omega}_1$.

Lemma 1. *For any given feasible ξ, if $S-(C+E\widehat{C})+C_3-C_2(T-t_1) \leq 0$, the optimal solution occurs at the boundary of $\mathbf{\Omega}_1$. In addition, the optimal solution is unique.*

In contrast to Lemma 1, the following lemma shows that there is a unique interior maximizer of $\Pi\,(t_1, T, \xi)$ on $\mathbf{\Omega}_2$.

Lemma 2. *For any given feasible ξ, if $S - (C + E\widehat{C}) + C_3 - C_2(T - t_1) > 0$, there is a unique interior maximizer of $\Pi\,(t_1, T, \xi)$ on $\mathbf{\Omega}_2$.*

We can now combine the results proven in the aforementioned lemmas, and state the following theorem that establishes the existence and uniqueness of solution to the problem.

Theorem 1. *For any given feasible ξ, there is a unique interior maximizer for $\Pi\,(t_1, T, \xi)$ on $\mathbf{\Omega}$.*

Corollary 1. *Let $\Pi^{\text{WD}}(t_1, T)$ be the model without deterioration (i.e., $\theta = 0$ and $\xi = 0$), the maximizer of $\Pi^{\text{WD}}(t_1, T)$, denoted by $(t_1^{\text{WD}}, T^{\text{WD}})$, not only exists but is unique.*

Corollary 2. *Let $\Pi^{\text{WP}}(t_1, T)$ be the model without preservation technology investment (i.e., $\xi = 0$), the maximizer of $\Pi^{\text{WP}}(t_1, T)$, denoted by $(t_1^{\text{WP}}, T^{\text{WP}})$, not only exists but is unique.*

Corollary 3. *Let $\Pi^{\text{WC}}(t_1, T, \xi)$ be the model without carbon emission constrain (i.e., $E = 0$), then for a given ξ, the maximizer of $\Pi^{\text{WC}}(t_1, T, \xi)$, denoted by $(t_1^{\text{WC}}, T^{\text{WC}})$, not only exists but is unique.*

Having proved the existence and uniqueness of the optimal replenishment strategy for a given preservation technology investment, we now focus on the preservation technology investment decision that maximizes retailer's profit. The purpose of retailer investing in preservation technology is to raise its profit, and thus it is clear to see that $\Pi^{\text{WP}}(t_1, T) \leq \Pi(t_1, T, \xi) < \Pi^{\text{WD}}(t_1, T)$. Let $\overline{\xi} \equiv \min\{w, \Pi^{\text{WD}}(t_1^{\text{WD}}, T^{\text{WD}}) - \Pi^{\text{WP}}(t_1^{\text{WP}}, T^{\text{WP}})\}$, then $\overline{\xi}$ is the upper bound of the optimal preservation technology investment. And hence, the objective can be written as $\max_{0 \leq \xi \leq \overline{\xi}} \max_{(t_1, T) \in \Omega} \Pi(t_1, T, \xi)$. From Theorem 1, since the optimal values of t_1 and T can be uniquely determined for a given ξ, t_1 and T can be seen as a function of ξ. Let $\Pi^*(t_1(\xi), T(\xi), \xi)$ denote the maximum value function of Π for a given ξ, then the problem becomes $\max_{0 \leq \xi \leq \overline{\xi}} \Pi^*(t_1(\xi), T(\xi), \xi)$. By Berge's Maximum Theorem, because $\Pi^*(t_1(\xi), T(\xi), \xi)$ is continuous in ξ on the interval $[0, \overline{\xi}]$, Weierstrass's Theorem can be asserted to prove the existence of a maximum. If there is an interior solution for ξ^*, a direct application of the Envelope Theorem gives the optimality condition as $\frac{d\Pi(t_1(\xi), T(\xi), \xi)}{d\xi} = \frac{\partial \Pi(t_1, T, \xi)}{\partial \xi} = 0$.

In order to solve the problem numerically using a fairly iterative search algorithm, we prove the concavity of total profit per unit time in terms of preservation technology investment under some mild assumptions. We formally establish this result in the following theorem.

Theorem 2. *For any given feasible (t_1, T), if the productivity of invested capital, $m(\xi)$, is a strictly concave function of ξ (i.e., $m'(\xi) > 0$ and $m''(\xi) < 0$ or diminishing marginal productivity of invested capital), the total profit per unit time, $\Pi(t_1, T, \xi)$, is a strictly concave function with respect to ξ.*

Theorem 2 indicates that there exists a unique value of $\xi \in \left[0, \overline{\xi}\right]$ which maximizes $\Pi(t_1, T, \xi)$ for any given t_1 and T. Combining Theorems 1 and 2, we give a fairly iterative search algorithm to obtain the local maximum for the problem.

Algorithm 1.

Step 1 Start with $k = 0$ and choose the initial trial value ξ_0, where $0 \le \xi_0 \le \overline{\xi}$.

Step 2 For a given preservation technology investment ξ_k, find the optimal values of t_1 and T, denoted by $t_{1,k}$ and T_k, for $\Pi(t_1, T, \xi_k)$ from Eqs. (3.6) and (3.7).

Step 3 Use the result gained from Step 2, and then determine the optimal value of ξ, denoted by ξ_{k+1}, for $\Pi(t_{1,k}, T_k, \xi)$ by (3.8).

Step 4 If the difference between ξ_{k+1} and ξ_k is sufficiently small (e.g., $|\xi_k - \xi_{k+1}| < 0.00005$), set $\xi^* = \xi_k$, then $(t_1^*, T^*, \xi^*) = (t_{1,k}^*, T_k^*, \xi_k^*)$ is the optimal solution and stop. Otherwise, set $k = k + 1$ and return to Step 2.

To begin the search, we need a starting value for ξ. Note that ξ is bounded between 0 and $\overline{\xi}$, we might choose $\xi_0 = \frac{\overline{\xi}}{2}$ for our initial guess in Step 1. In Step 2 where Theorem 1 is applied, there exists a unique local maximum solution $(t_{1,k}^*, T_k^*)$ of $\Pi(t_1, T, \xi_k)$ for current ξ_k. Then the value of $\Pi(t_{1,k}^*, T_k^*, \xi_{k+1}^*)$ is improving in Step 3 where Theorem 2 is applied. Further, by using the results in Theorems 1 and 2, it is straightforward to see that $\Pi(t_{1,k}^*, T_k^*, \xi_{k+1}^*) > \Pi(t_{1,k}^*, T_k^*, \xi_k^*)$ and $\Pi(t_{1,k}^*, T_k^*, \xi_k^*) > \Pi(t_{1,k-1}^*, T_{k-1}^*, \xi_k^*)$, and hence $\Pi(t_{1,k}^*, T_k^*, \xi_k^*)$ is a monotone increasing sequence. By Monotone Convergence Theorem, the procedure repeating Steps 2 and 3 would converge to a local maximum of $\Pi(t_1, T, \xi)$ because $\Pi(t_1, T, \xi)$ is bounded between $\Pi^{\text{WP}}(t_1^{\text{WP}}, T^{\text{WP}})$ and $\Pi^{\text{WD}}(t_1^{\text{WD}}, T^{\text{WD}})$. Although we cannot show the concavity property of $\Pi(t_1, T, \xi)$, the algorithm can be repeated to identify the global maximum solution by using several starting values of ξ.

4. Sensitivity to Carbon Cost Parameters

In this section, we discuss how changes in various problem parameters affect the optimal replenishment strategy and carbon emissions per unit time. We begin the section with sensitivity analyses of preservation technology investment to the optimal service level. Recall that we have shown that there is a unique interior global maximizer for $\Pi(t_1, T, \xi)$ in Theorem 1. Hence, the optimal values of t_1 and T can be represented as functions of problem parameters. Furthermore, we also observe that the optimal solution of $\Pi(t_1, T, \xi)$ always lies on the interior of $\mathbf{\Omega}_2$; and therefore, $\mathbf{\Omega}_1$ can be excluded from further consideration. From now on in this section, we assume both $\beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C})\,[\beta + \theta - m(\xi)] < 0$ and $S - (C + E\widehat{C}) + C_3 - C_2(T - t_1) > 0$ hold. The proposition below characterizes the behavior of optimal service level with respect to changes in the chosen preservation technology investment.

Proposition 1. *The optimal service level is strictly increasing in preservation technology investment.*

Proposition 1 shows that the optimal service level increases strictly in preservation technology investment. The result of the proposition is intuitive. If the retailer increases its preservation technology investment, the deterioration rate for the item would decrease and the duration of the inventory holding period becomes larger, and so the optimal service level rises.

In the next result we state the effects of carbon price on the optimal carbon emissions per unit time and total profit per unit time.

Proposition 2. *For any given feasible carbon cap and preservation technology investment, we have the following results:*

(1) *The optimal carbon emissions per unit time is strictly decreasing in carbon price.*

(2) *The optimal total profit per unit time is strictly pseudoconvex in carbon price.*

Proposition 2 indicates that increasing carbon price can reduce the carbon emissions efficiently. Since a higher carbon price will increase the overall cost of operating the inventory system, adding the carbon price can significantly induce the retailer to reduce carbon purchasing cost to maximize its own profit. However, when the carbon price is sufficiently high, since the retailer can sell its carbon allowances to earn more profit from the carbon trading market, total profit per unit time increases strictly as carbon price increases. Hence, the optimal total profit per unit time is pseudoconvex in in carbon price.

In the last of this section, we study the effect of changing the carbon cap on the optimal replenishment strategy. The following proposition summarizes the outcomes.

Proposition 3. *For any given feasible carbon price and preservation technology investment, if the carbon cap increases, the optimal replenishment strategy and carbon emissions per unit time remain constant, but the optimal total profit per unit time increases linearly.*

Proposition 3 indicates that both the optimal replenishment strategy and carbon emissions per unit time are not affected by carbon cap. The inefficiency of carbon cap on reducing carbon emissions is because the retailer can buy or sell its carbon credits with a fixed carbon price on the carbon trading market. Furthermore, since the retailer can sell its allowances to earn more profit from the carbon trading market, the total profit per unit time will increase linearly as carbon cap increases.

5. Extension to the Model with the Carbon Offset Policy

In this section, we extend the above results to analyze the model with the Carbon Offset policy. The Carbon Offset policy allows individuals and businesses to compensate for their carbon emissions by funding a project or activity around the world that reduces or stores greenhouse gases. The project might involve rolling out clean energy technologies or soaking up CO2 directly from the air through the planting of trees. Therefore, the retailer can reduce its carbon emissions without actually polluting less under the Carbon Offset policy. In other words, it provides the retailer an opportunity to offset its carbon emissions by supporting projects that reduce carbon emissions elsewhere with unit price E. Hence, the retailer's total profit per unit time, denoted by $\Pi^{\mathrm{co}}(t_1, T, \xi)$, can be represented as

$$\Pi^{\mathrm{co}}(t_1, T, \xi) = \frac{TP(t_1, T, \xi)}{T} - E \times \max\left\{\frac{CE(t_1, T, \xi)}{T} - \varpi, 0\right\}. \tag{5.1}$$

In order to analyze the problem clearly, for a given preservation technology investment, we divide the problem into two subproblems:

Case 1:

$$\max_{t_1,T} \quad \frac{TP(t_1,T,\xi)}{T}, \text{ s.t. } T > t_1 > 0 \text{ and } \frac{CE(t_1,T,\xi)}{T} - \varpi \leq 0,$$

and

Case 2:

$$\max_{t_1,T} \quad \frac{TP(t_1,T,\xi)}{T} - E \times \left[\frac{CE(t_1,T,\xi)}{T} - \varpi\right],$$
$$\text{s.t. } T > t_1 > 0 \text{ and } \frac{CE(t_1,T,\xi)}{T} - \varpi > 0.$$

In Case 1, the amount of retailer's carbon emitted is constrained to be less than certain cap ϖ. It is also known as Mandatory Carbon Emissions Capacity policy. On the other hand, the retailer should pay for its excess emissions with unit price E in Case 2, which can be seen as the model with the Carbon Cap-and-Trade policy. Then the optimal total profit per unit time for the model with the Carbon Offset policy is to select the minimum between Case 1 and Case 2. We next study the effects of carbon parameters on the optimal replenishment strategy for each case. The following two propositions state and prove these results formally.

Proposition 4. *In Case* 1*, for a given preservation technology investment, then:*

(1) *There exists a unique global maximum for the retailer's total profit per unit time.*

(2) *The optimal carbon emissions per unit time first increases strictly, then remains constant as carbon cap increases.*

(3) *The optimal total profit per unit time first increases strictly, and then remains constant as carbon cap increases.*

Proposition 5. *In Case* 2*, for a given preservation technology investment, then:*

(1) *There exists a unique global maximum for the retailer's total profit per unit time.*

(2) *The optimal carbon emissions per unit time first decreases strictly, then remains constant as carbon price increases.*

(3) *The optimal carbon emissions per unit time first remains constant, then increases linearly as carbon cap increases.*

(4) *The optimal total profit per unit time first decreases strictly, then remains constant as carbon price increases.*

(5) *The optimal total profit per unit time first increases strictly, then decreases strictly as carbon cap increases.*

Using the results from Propositions 4 and 5, we can now analyze the optimal replenishment strategy for the model with the Carbon Offset policy among in the Cases 1 an 2. Let ϖ^{CT} denote the optimal carbon emissions per unit time with the Carbon

Cap-and-Trade policy for a given carbon price E. For any given feasible preservation technology investment, the following proposition summarizes the solution to the decision problem the retailer faces.

Proposition 6. *For any given feasible preservation technology investment, the optimal replenishment strategy is Case* 1 *if* $\varpi > \varpi^{\mathrm{CT}}$*; otherwise the optimal replenishment strategy is Case* 2*.*

Let ϖ^{WC} denote the optimal carbon emissions per unit time without carbon emission constrain. Proposition 7 below characterizes the effects of carbon parameters on the optimal replenishment strategy under the Carbon Offset policy without proof as it follows immediately from Propositions 2, 3, and 4–6.

Proposition 7. *In the model with the Carbon Offset policy, for a given preservation technology investment, then:*

(1) *There exists a unique global maximum for the retailer's total profit per unit time.*

(2) *The optimal carbon emissions per unit time first remains constant, then increases strictly and finally remains constant as carbon cap increases.*

(3) *The optimal total profit per unit time first increases strictly, then remains constant as carbon cap increases.*

(4) *If* $\varpi > \varpi^{\mathrm{WC}}$*, the optimal carbon emissions per unit time remains constant as carbon price increases; otherwise the optimal carbon emissions per unit time first decreases strictly, then remains constant as carbon price increase.*

(5) *The optimal total profit per unit time remains constant as carbon price increases if* $\varpi > \varpi^{\mathrm{WC}}$*; otherwise the optimal total profit per unit time first decreases strictly, then remains constant as carbon price increase.*

Further, as a consequence of Propositions 4–7 we have the following remarks in the trivial case.

Remark 1. Then the model reduces to the case of Carbon Tax policy if $\varpi = 0$. On other hand, the model reduces to the case of Mandatory Carbon Emissions Capacity policy (i.e., Case 1) if E is sufficiently large.

Remark 2. For any given feasible preservation technology investment, the optimal total profit per unit time for the Carbon Cap-and-Trade policy is greater or equal to than the optimal total profit per unit time for the Carbon Offset policy.

6. Numerical Example and Sensitivity Analysis

In this section, we illustrate the proposed model with a example. Consider an inventory situation where $\alpha = 1000$, $\beta = 0.2$, $A = 120$, $S = 35$, $C = 20$, $C_1 = 3$, $C_2 = 6$, $C_3 = 10$, $\widehat{A} = 200$, $\widehat{C} = 5$, $\widehat{C}_1 = 2.5$, $E = 0.2$, $\varpi = 6250$, $\theta = 0.2$, $w = 500$, $\beta(x) = e^{-x}$

and $m(\xi) = (1 - e^{-0.01\xi})\theta$. We first compute the optimal values of $\Pi^{\text{WD}}(t_1, T)$ and $\Pi^{\text{WP}}(t_1, T, 0)$ as $\Pi^{\text{WD}}(0.6220, 0.6774) = 14786.4$ and $\Pi^{\text{WP}}(0.1966, 0.3171, 0) = 14247.8$, respectively. And hence the upper bound for ξ gives $\overline{\xi} = \min\{500, 538.5765\} = 500$. By applying Algorithm 1 with starting initial $\xi_0 = \frac{\overline{\xi}}{2} = 250$ yields $\Pi(t_1^*, T^*, \xi^*) = \Pi(0.4733, 0.5429, 220.3577) = 14448.9$ after 13 iterations, and so the corresponding optimal carbon emissions per unit time is $\frac{CE(t_1^*,T^*,\xi^*)}{T^*} = 6137.93$. In addition, by Proposition 6, since $\varpi = 6250 > 6137.93 = \varpi^{\text{CT}}$, we can then obtain the optimal total profit per unit time and corresponding carbon emissions per unit time for the model with the Carbon Offset policy as $\Pi^{\text{CO}}(t_1^*, T^*, \xi^*) = \Pi(0.5417, 5985, 236.1339) = 14444.3$ and $\frac{CE(t_1^*,T^*,\xi^*)}{T^*} = 6250$. Figure 1 demonstrates the impact of carbon cap on the total profit per unit time for the different carbon regulations. It is straightforward to see that the total profit per unit with Carbon Cap-and-Trade policy increases as carbon cap increases. Meanwhile, the total profit per unit without carbon emission constrain is always greater or equal to that with Carbon Offset policy.

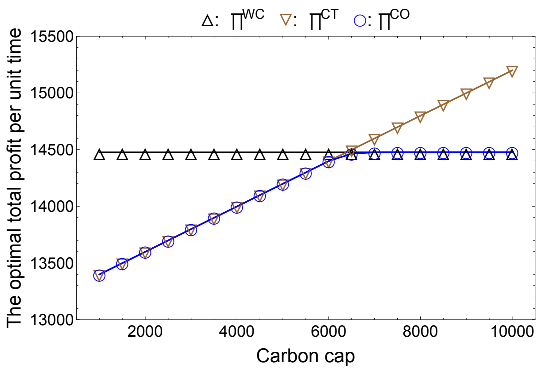
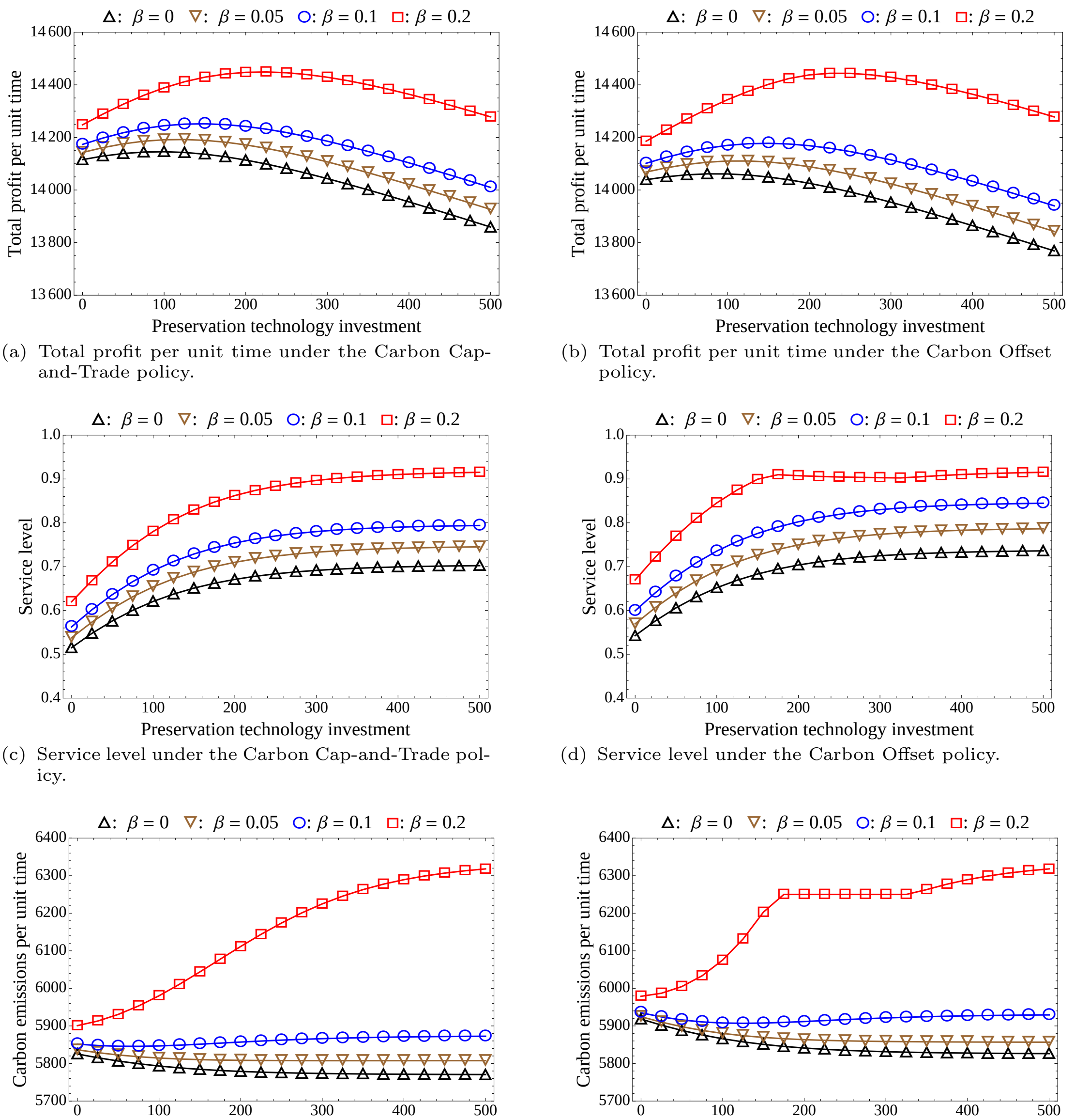


Figure 1: Impacts of carbon cap on the total profit per unit time.

Further, in order to ensure whether the solutions for the both model are the global maximum or not, we run the numerical results with distinct starting values of $\xi_0 = 0, \frac{\overline{\xi}}{20}, \frac{2\overline{\xi}}{20}, \ldots, \overline{\xi}$, which are summarized in the graphical presentation. Figures 2(a) and 2(b) reveal that the total profit per unit time for both policies are unimodal in preservation technology technology. And thus, the local maximum obtained here from the proposed algorithm are indeed the global maximum. Moreover, we also observe from Figures 2(c) and 2(d) that increasing the value of preservation technology technology can efficiently give rise to an increasing in service level. Figures 2(e) and 2(f) indicate that increasing the preservation technology investment results in a decreasing in the carbon emissions per unit time only when β is low. When β is high enough, it represents that higher inventories stimulate greater demand and thereby induce a higher order quantity from the retailer. Therefore, the carbon emissions generated from operating the inventory system increases.

Next, we perform some numerical studies to investigate the influences of carbon price and carbon cap on the retailer's optimal replenishment and preservation technology



(a) Total profit per unit time under the Carbon Cap-and-Trade policy.

(b) Total profit per unit time under the Carbon Offset policy.

(c) Service level under the Carbon Cap-and-Trade policy.

(d) Service level under the Carbon Offset policy.

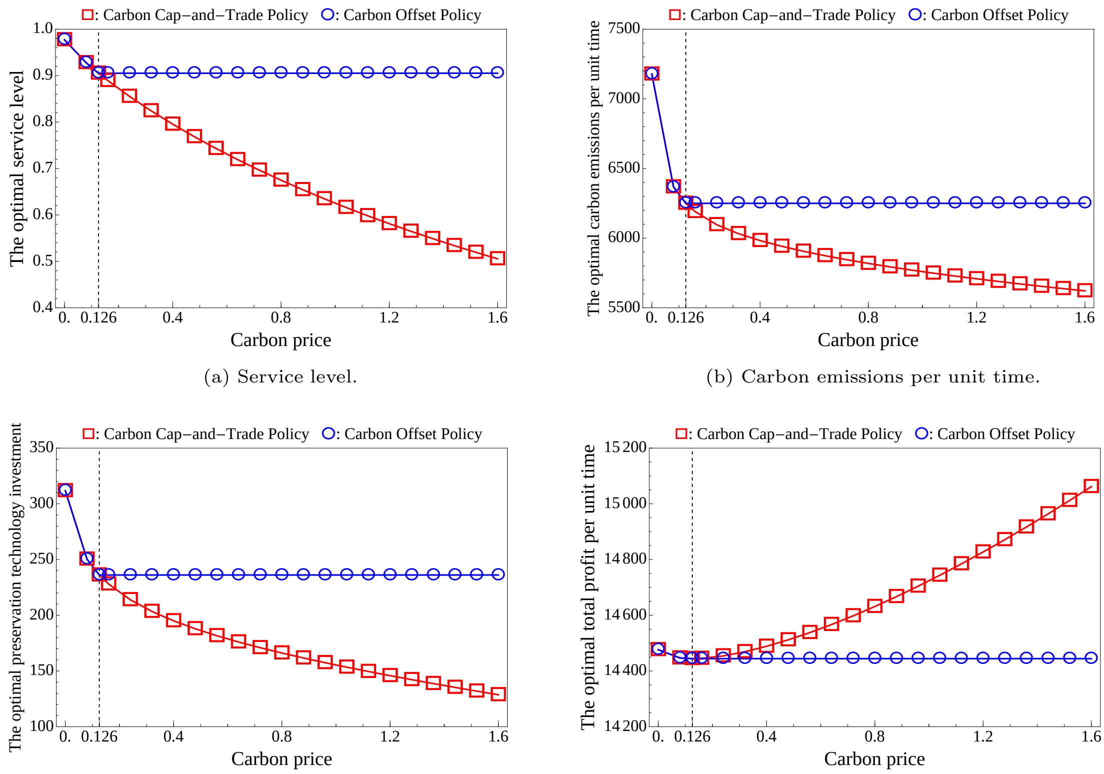
(e) Carbon emissions per unit time under the Carbon Cap-and-Trade policy.

(f) Carbon emissions per unit time under the Carbon Offset policy.

Figure 2: Impacts of preservation technology investment on service level and total profit per unit time.

investment strategies. In each of the two carbon parameters we choose $\beta = 0.05$ and 0.2. In the first study, we examine the effect of carbon price on the retailer's optimal service level, preservation technology investment, carbon emissions per unit time and total profit per unit time. From Figures 5 and 4, we first observe that the optimal service level and carbon emissions per unit time are both decreasing in carbon price for the two policies.

These happen because higher carbon price increases the unit inventory purchasing and holding costs, the retailer would reduce its service level and purchasing cost to maximize profit. When the carbon price is high, the reduced service level and purchasing cost consequently result in decreased optimal preservation technology investment. However, when β and carbon price are low, we find that increasing the carbon price leads to an increasing in the optimal preservation technology investment. This is because, under such circumstances, the retailer can increases its the optimal total profit per unit time by raising its preservation technology investment to reduce deterioration loss. Finally, Figures 5 and 4 show that the optimal total profit per unit time first decreases and then increases with increasing carbon price because the retailer can sell its carbon credits to earn more profit from the carbon trading market under the Carbon Cap-and-Trade policy. However, for the case of Carbon Offset policy, since it is not profitable to buy extra emission rights when carbon price is large, the total optimal total profit per unit time first decreases, and then remains constant.



(a) Service level.

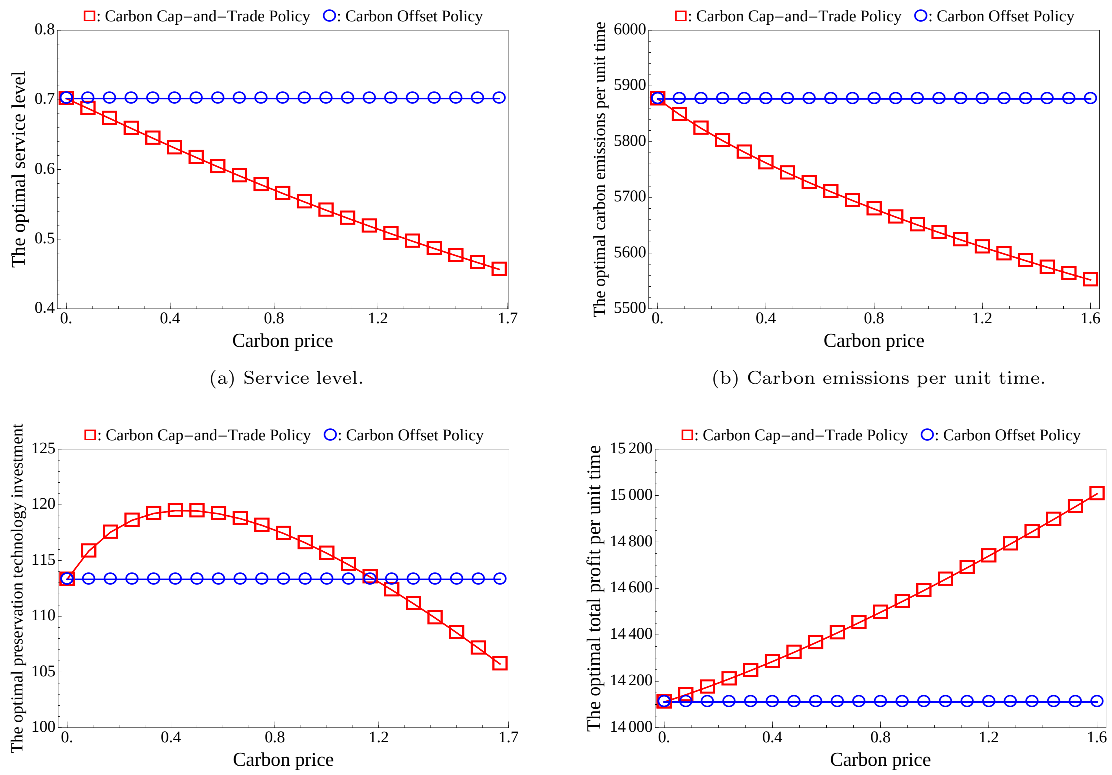
(b) Carbon emissions per unit time.

(c) Preservation technology investment.

(d) Total profit per unit time.

Figure 3: Impacts of carbon price on the optimal service level, preservation technology investment, carbon emissions per unit time and total profit per unit time when $\beta = 0.2$.

In the second study, we further investigated the effect of carbon cap on the retailer's optimal replenishment and preservation technology investment strategies. For the Carbon Cap-and-Trade policy, since the retailer can buy and sell its carbon credits on the



(a) Service level.

(b) Carbon emissions per unit time.

(c) Preservation technology investment.

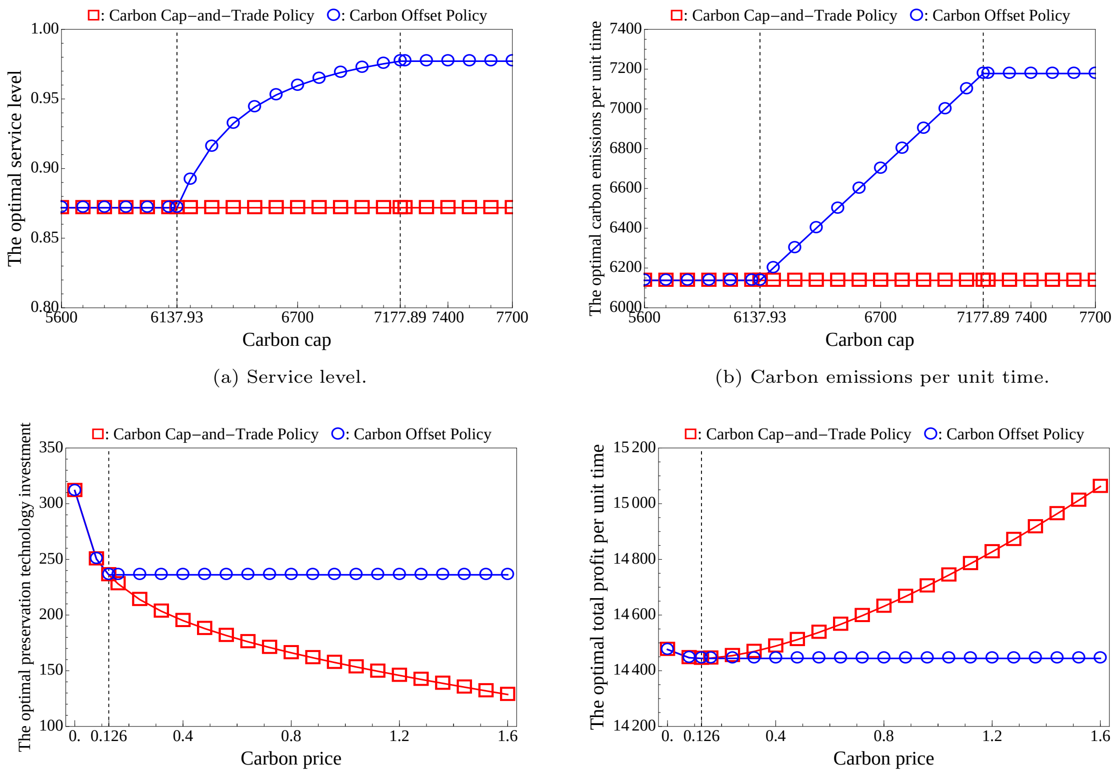
(d) Total profit per unit time.

Figure 4: Impacts of carbon price on the optimal service level, preservation technology investment, carbon emissions per unit time and total profit per unit time when $\beta = 0.05$.

carbon trading market, both the optimal replenishment strategy and carbon emissions per unit time are not affected by carbon cap. Hence, from Figures 6 and 7, we can find that the optimal service level, carbon emissions per unit time and preservation technology investment remain constant, but the total profit per unit time increases linearly as carbon cap increases.

On the other hand, for the Carbon Offset policy, we first observe that the optimal service level and carbon emissions per unit time first remain constant, then increase strictly and finally remains constant as carbon cap increases. These happen because of the following reasons. If $\varpi \geq \varpi^{\mathrm{WC}}$, then the constraint on the amount of carbon emitted is non-binding. If $\varpi \leq \varpi^{\mathrm{CT}}$, since the shadow price is greater than its unit carbon cost, the retailer is willing to pay for an extra unit of emission rights until its amount of carbon emitted equals ϖ^{CT}. Hence, the optimal service level and carbon emissions per unit time remain constant as carbon cap increases. However, the retailer needs prepare more inventory when carbon cap increases if $\varpi^{\mathrm{CT}} < \varpi < \varpi^{\mathrm{WC}}$ because the constraint on the amount of carbon emitted is binding. Therefore, both the optimal service level and carbon emissions per unit time increase strictly as carbon cap increases. Furthermore, from above arguments, because the constraint on the amount of carbon emitted is binding when $\varpi < \varpi^{\mathrm{WC}}$, the total profit per unit time first increases strictly, and then remains constant as carbon cap increases.

Finally, when β is high, we notice from Figure 5(c) that the optimal preservation technology investment increases strictly as carbon cap increases when $\varpi^{\mathrm{CT}} < \varpi < \varpi^{\mathrm{WC}}$. This happens because high inventory level induces more demand when β is high, the retailer needs to prepare more inventory to satisfy customers' orders as carbon cap increases. The increased inventory leads the retailer to increase its preservation technology investment to reduce the deterioration loss. Conversely, from Figure 6(c), we find that the optimal preservation technology investment decreases strictly as carbon cap increases when β is low and $\varpi^{\mathrm{CT}} < \varpi < \varpi^{\mathrm{WC}}$. This is because that the effect of inventory level on demand is limit when β is low, the retailer does not need to prepare more inventory as carbon cap increases. Instead, the increasing of carbon cap reduces the pressure of deterioration loss, and therefore the optimal preservation technology investment decreases.



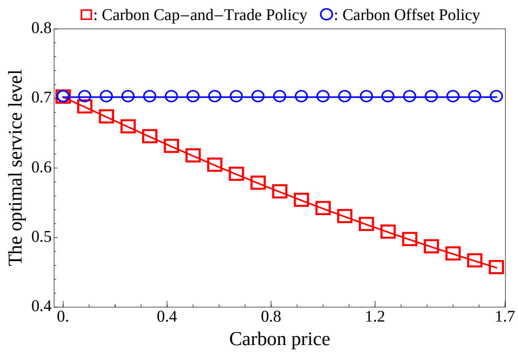
(a) Service level.

(b) Carbon emissions per unit time.

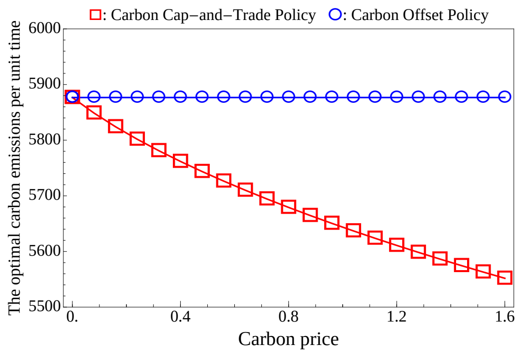
(c) Preservation technology investment.

(d) Total profit per unit time.

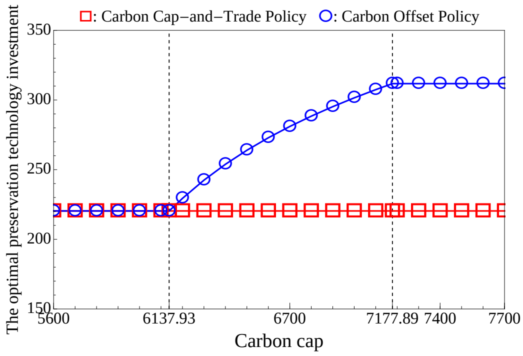
Figure 5: Impacts of carbon price on the optimal service level, preservation technology investment, carbon emissions per unit time and total profit per unit time when $\beta = 0.2$.



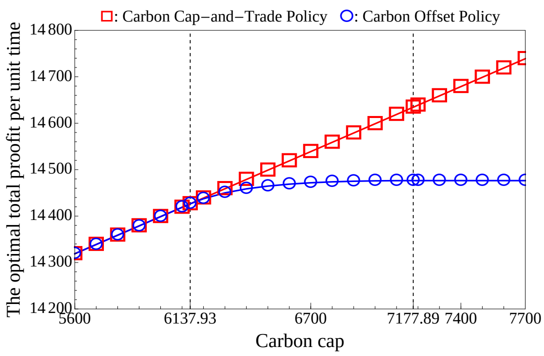
(a) Service level.



(b) Carbon emissions per unit time.



(c) Preservation technology investment.

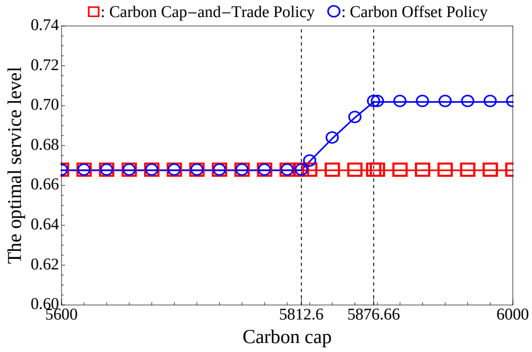


(d) Total profit per unit time.

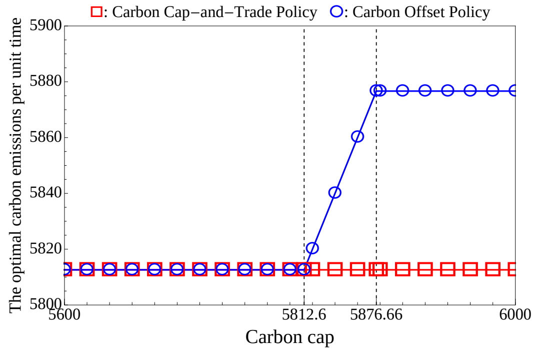
Figure 6: Impacts of carbon cap on the optimal service level, preservation technology investment, carbon emissions per unit time and total profit per unit time when $\beta = 0.2$.

7. Concluding Remarks and Future Work

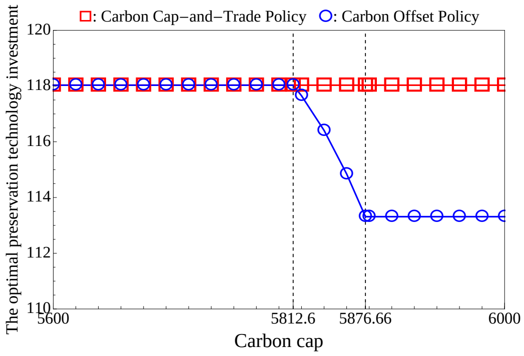
Since many countries or regions have made either voluntary or regulatory efforts to reduce their carbon emissions, more and more companies are realizing carbon as a key consideration in business and investment decision making activities. At present, fossil fuels are the dominant energy sources of the global primary energy supply, and will likely remain so for the rest of the century. Because carbon is the basic element in fossil energy, cutting carbon equals to cost savings and operational efficiency. In this paper, an analytical model is developed to assess the impacts of preservation technology investment and carbon cost parameters on inventory replenishment decision making and the implications for retailer in a competitive market environment. The analytical formulations of the model are shown and the theoretical results are discussed and compared. The numerical result is further used to illustrate how the retailer demands benefit from preservation technology investment under various carbon emission regulatory policies. It is found that the retailer can improve the customer service level thorough efficient investment in preservation technology. However, the reduction of carbon emissions can be achieved by increasing preservation technology investment only when the motivational



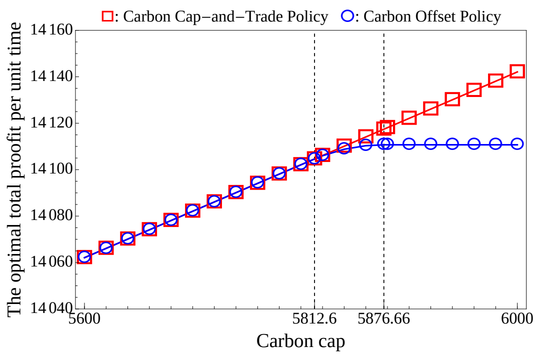
(a) Service level.



(b) Carbon emissions per unit time.



(c) Preservation technology investment.



(d) Total profit per unit time.

Figure 7: Impacts of carbon cap on the optimal service level, preservation technology investment, carbon emissions per unit time and total profit per unit time when $\beta = 0.05$.

effect of inventory on demand (i.e., β) is low. When the motivational effect of inventory on demand is high enough, higher inventories stimulate greater demand. Hence the focus of retailer is centered on making more profit, not on reducing the loss of deterioration and carbon emissions. Moreover, the sensitivity analysis results also reveal that the policymakers can reduce retailer's carbon emissions by adjusting carbon price and carbon cap accordingly, and the motivational effect of inventory on demand has a significant effect on preservation technology investment decision.

In the future, we may extend this work to consider some inventory related problems. In this study, since the selling price of product is held constant, we can incorporate pricing into this work. The use of endogenous price would provide us with an opportunity to view the pricing decision of retailers conjointly with their inventory decision. Also, we could extend the the model with time-varying demand or stochastic demand over a finite planning horizon. In addition, delays in product availability are common in real-world scenario, hence the stockout compensation policy may be incorporated into the presented model to improve market efficiency and increase the retailers sales and profit. Finally, since profit, service level and carbon emissions are conflicting to each other, multi-objective optimization may be employed to extend the presented model further.

Acknowledgements

The authors would like to thank the editor and anonymous reviewers for their valuable and constructive comments, which have led to a significant improvement in the manuscript.

Appendix

Proof of Lemma 1. The Karush-Kuhn-Tucker conditions for the optimization problem are

$$\frac{\partial \Pi(t_1,T,\xi)}{\partial t_1} - \lambda C_2 = 0,$$

$$\frac{\partial \Pi(t_1,T,\xi)}{\partial T} + \lambda C_2 = 0,$$

$$S-(C+E\widehat{C})+C_3-C_2(T-t_1) \le 0,$$

$$\lambda[S-(C+E\widehat{C})+C_3-C_2(T-t_1)] = 0,$$

$$\lambda \ge 0,$$

where λ is Lagrange multiplier. Since $\Pi(t_1,T,\xi) > 0$ on $\mathbf{\Omega}_1$, if $S-(C+E\widehat{C})+C_3-C_2\,(T-t_1) \le 0$, it is straightforward to see that $\frac{\partial \Pi(t_1,T,\xi)}{\partial T} < 0$ from (3.7), which implies that $\lambda > 0$. Therefore by Complementary Slackness theorem, we have $S-(C+E\widehat{C})+C_3-C_2(T-t_1) = 0$. That is, the optimal solution always lies on the constraint boundary of $\mathbf{\Omega}_1$.

Next, we show that the optimal solution is unique. Since the optimal solution always lies on the constraint boundary of $\mathbf{\Omega}_1$, rearranging the inequality yields $T = t_1 + \frac{S-(C+E\widehat{C})+C_3}{C_2}$. Substituting the result into $f\,(t_1,T,\xi)$ and taking the second derivative of $f\,(t_1,T,\xi)$ with respective of t_1, we then have

$$\begin{aligned}&\frac{\partial^2 f\left(t_1, t_1+\frac{S-(C+E\widehat{C})+C_3}{C_2}, \xi\right)}{\partial t_1^2}\\ &= \left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\,[\beta+\theta-m(\xi)]\right\} e^{[\beta+\theta-m(\xi)]t_1}\\ &< 0.\end{aligned}$$

Since $f\left(t_1, t_1+\frac{S-(C+E\widehat{C})+C_3}{C_2}, \xi\right)$ is strictly concave in t_1 and $t_1+\frac{S-(C+E\widehat{C})+C_3}{C_2}$ is affine, then $\Pi\left(t_1, t_1+\frac{S-(C+E\widehat{C})+C_3}{C_2}, \xi\right)$ is strictly pseudoconcave in t_1.
Thus, $\Pi\left(t_1, t_1+\frac{S-(C+E\widehat{C})+C_3}{C_2}, \xi\right)$ attains a local maximum on $\mathbf{\Omega}_1$, which is the unique global maximum due to the strict pseudoconcavity of $\Pi\left(t_1, t_1+\frac{S-(C+E\widehat{C})+C_3}{C_2}, \xi\right)$. This completes the proof of Lemma 1. □

Proof of Lemma 2. We first show that the maximizer of $\Pi(t_1, T, \xi)$ on $\mathbf{\Omega}_2$ is unique. Taking the second partial derivatives of $f(t_1, T, \xi)$ with respect to t_1 and T gives

$$\begin{aligned}\frac{\partial^2 f(t_1,T,\xi)}{\partial T^2} &= -\alpha\left\{C_2 b(T-t_1) - [S-(C+E\widehat{C})+C_3-C_2(T-t_1)]b'(T-t_1)\right\} \\ &< 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f(t_1,T,\xi)}{\partial t_1^2} &= \alpha\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\}e^{[\beta+\theta-m(\xi)]t_1} \\ &\quad -\alpha\left\{C_2 b(T-t_1)-[S-(C+E\widehat{C})+C_3-C_2(T-t_1)]b'(T-t_1)\right\} \\ &= \alpha\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\}e^{[\beta+\theta-m(\xi)]t_1} \\ &\quad +\frac{\partial^2 f(t_1,T)}{\partial T^2} \\ &< 0,\end{aligned}$$

and

$$\frac{\partial^2 f(t_1,T,\xi)}{\partial t_1 \partial T} = -\frac{\partial^2 f(t_1,T)}{\partial T^2}.$$

The corresponding determinant of the Hessian matrix is then given by

$$\begin{aligned}|\mathbf{H}| &= \frac{\partial^2 f(t_1,T,\xi)}{\partial T^2}\times\frac{\partial^2 f(t_1,T,\xi)}{\partial t_1^2}-\left[\frac{\partial^2 f(t_1,T,\xi)}{\partial t_1\partial T}\right]^2 \\ &= \alpha\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\}e^{[\beta+\theta-m(\xi)]t_1}\frac{\partial^2 f(t_1,T,\xi)}{\partial T^2} \\ &> 0.\end{aligned}$$

Thus, $f(t_1, T, \xi)$ is a strictly concave in (t_1, T) on $\mathbf{\Omega}_2$. Further, since T is affine, then $\Pi(t_1, T, \xi)$ is strictly pseudoconcave in (t_1, T) on $\mathbf{\Omega}_2$. Because $T-t_1$ and $S-(C+E\widehat{C})+C_3-C_2(T-t_1)$ are linear and $-f(t_1, T, \xi)$ is strictly convex, this implies that $\mathbf{\Omega}_2$ is a convex set. Therefore, by concave fractional programming, there exists a unique solution such that $\Pi(t_1, T, \xi)$ is maximum on $\mathbf{\Omega}_2$.

We then show that the optimal maximizer is an interior point of $\mathbf{\Omega}_2$. For any given $t_1 > 0$, we observe that (3.6) holds if and only if $T > t_1$. In addition, from (3.7), it is straightforward to see that the the $\Pi(t_1, T, \xi) > 0$ if and only if $S-(C+E\widehat{C})+C_3-C_2(T-t_1) > 0$. Combining the arguments above sufficiently guarantees that $\Pi(t_1, T, \xi)$ has a unique interior maximizer on Ω_2, which also completes the proof of Lemma 2. □

Proof of Theorem 1. For convenience, let $\Pi_1(t_1, T, \xi)$ and $\Pi_2(t_1, T, \xi)$ denote the total profit per unit time on the corresponding respectively feasible region $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$. Since $\Pi(t_1, T, \xi)$ is continuous on Ω, it is straightforward to verified that

$$\max_{(t_1,T)\in\mathbf{\Omega}_2}\Pi_2(t_1,T,\xi) > \max_{(t_1,T)\in\mathbf{\Omega}_2}\Pi_2\left(t_1, t_1+\frac{S-(C+E\widehat{C})+C_3}{C_2},\xi\right)$$

$$
\begin{aligned}
&= \max_{(t_1,T)\in\mathbf{\Omega}_1} \Pi_1\left(t_1, t_1 + \frac{S-(C+E\widehat{C})+C_3}{C_2}, \xi\right)\\
&= \max_{(t_1,T)\in\mathbf{\Omega}_1} \Pi_1(t_1,T,\xi).
\end{aligned}
$$

Hence, $\Pi\,(t_1,T,\xi)$ has a unique interior maximizer on Ω and the optimal values of t_1 and T can be uniquely determined by these first-order conditions (3.6) and (3.7). This completes the proof of Theorem 1. □

Proof of Theorem 2. Using the result derived in (3.8) and then taking the second derivative of $\Pi(t_1,T,\xi)$ with respect to ξ yields

$$
\begin{aligned}
\frac{\partial^2\Pi(t_1,T,\xi)}{\partial\xi^2} = &-\frac{\alpha m''(\xi)}{T[\beta+\theta-m(\xi)]^3}\\
&\times\Bigg\{\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\,[\beta+\theta-m(\xi)]\right\}\\
&\times\sum_{n=3}^{\infty}\frac{(n-2)\,[\beta+\theta-m(\xi)]^n\,t_1^n}{n!}-(C+E\widehat{C})\sum_{n=2}^{\infty}\frac{[\beta+\theta-m(\xi)]^{n+1}\,t_1^n}{n!}\Bigg\}\\
&+\frac{\alpha\,[m'(\xi)]^2}{T[\beta+\theta-m(\xi)]^3}\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\,[\beta+\theta-m(\xi)]\right\}\\
&\times\sum_{n=4}^{\infty}\frac{(n-2)\,(n-3)\,[\beta+\theta-m(\xi)]^{n-1}\,t_1^n}{n!}.
\end{aligned}
$$

By using Assumption 3 and $m''(\xi)<0$, it is straightforward to see that $\frac{\partial^2\Pi(t_1,T,\xi)}{\partial\xi^2}<0$ implying that $\Pi\,(t_1,T,\xi)$ is strictly concave in ξ. □

Proof of Proposition 1. Differentiating implicitly on both sides of Eqs. (3.9) and (3.10) with respect to ξ and utilizing the facts that $\frac{\partial\Pi(t_1,T,\xi)}{\partial t_1}=0$ and $\frac{\partial\Pi(t_1,T,\xi)}{\partial T}=0$ yields

$$
\begin{aligned}
0 = &\frac{m'(\xi)}{[\beta+\theta-m(\xi)]^2}\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\,[\beta+\theta-m(\xi)]\right\}\\
&\times\left\{e^{[\beta+\theta-m(\xi)]t_1}-1-e^{[\beta+\theta-m(\xi)]t_1}\,[\beta+\theta-m(\xi)]\,t_1\right\}\\
&+\frac{dt_1}{d\xi}\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\,[\beta+\theta-m(\xi)]\right\}e^{t_1(\beta+\theta-m(\xi))}\\
&+m'(\xi)(C+E\widehat{C})\frac{e^{[\beta+\theta-m(\xi)]t_1}-1}{\beta+\theta-m(\xi)}+\left(\frac{dT}{d\xi}-\frac{dt_1}{d\xi}\right)\\
&\times\left\{C_2 b(T-t_1)-[S-(C+E\widehat{C})+C_3-C_2(T-t_1)]b'(T-t_1)\right\}
\end{aligned}
\tag{A.1}
$$

and

$$
-\frac{1}{\alpha}\left[\frac{\partial\Pi(t_1,T,\xi)}{\partial\xi}+1\right]=\left\{C_2 b(T-t_1)-[S-(C+E\widehat{C})+C_3-C_2(T-t_1)]b'(T-t_1)\right\}
$$

$$\times\left(\frac{dT}{d\xi}-\frac{dt_1}{d\xi}\right). \tag{A.2}$$

Our goal is to show that $\frac{d}{d\xi}\frac{t_1}{T}>0$. By using Taylor Series expansion on the exponent, (3.8) can be rewritten as

$$\begin{aligned}\frac{\partial\Pi(t_1,T,\xi)}{\partial\xi}+1=&-\frac{\alpha m'(\xi)}{T}\Bigg\{\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\}\\&\times\frac{e^{[\beta+\theta-m(\xi)]t_1}\left[\beta+\theta-m(\xi)\right]t_1+\left[\beta+\theta-m(\xi)\right]t_1-2e^{[\beta+\theta-m(\xi)]t_1}+2}{\left[\beta+\theta-m(\xi)\right]^3}\\&-(C+E\widehat{C})\frac{e^{[\beta+\theta-m(\xi)]t_1}-\left[\beta+\theta-m(\xi)\right]t_1-1}{\left[\beta+\theta-m(\xi)\right]^2}\Bigg\}.\\=&-\frac{\alpha m'(\xi)}{T[\beta+\theta-m(\xi)]^3}\Bigg\{\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\}\\&\times\sum_{n=3}^{\infty}\frac{(n-2)\left[\beta+\theta-m(\xi)\right]^n t_1^n}{n!}\\&-(C+E\widehat{C})\sum_{n=2}^{\infty}\frac{\left[\beta+\theta-m\left(\xi\right)\right]^{n+1}t_1^n}{n!}\Bigg\}.\end{aligned}$$

Since $m'(\xi)>0$, it is straightforward to show that $\frac{\partial\Pi(t_1,T,\xi)}{\partial\xi}+1>0$, which implies that (A.2) holds if and only if $\frac{dT}{d\xi}-\frac{dt_1}{d\xi}<0$.

Then substituting (3.8) into (A.1) and rearranging leads

$$0=g_1+g_2+\frac{dt_1}{d\xi}\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\}e^{t_1(\beta+\theta-m(\xi))},$$

where

$$\begin{aligned}g_1\equiv&\frac{m'(\xi)}{\left[\beta+\theta-m(\xi)\right]^2}\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\}\\&\times\left\{e^{[\beta+\theta-m(\xi)]t_1}-1-e^{[\beta+\theta-m(\xi)]t_1}\left[\beta+\theta-m(\xi)\right]t_1\right\}\\&+\frac{m'(\xi)}{\left[\beta+\theta-m(\xi)\right]^2}\left\{\beta S-(C_1+E\widehat{C}_1)-(C+E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\}\\&\times\frac{e^{[\beta+\theta-m(\xi)]t_1}\left[\beta+\theta-m(\xi)\right]t_1+\left[\beta+\theta-m(\xi)\right]t_1-2e^{[\beta+\theta-m(\xi)]t_1}+2}{\left[\beta+\theta-m(\xi)\right]T}\end{aligned}$$

and

$$\begin{aligned}g_2\equiv\,&m'(\xi)(C+E\widehat{C})\frac{e^{[\beta+\theta-m(\xi)]t_1}-1}{\beta+\theta-m\left(\xi\right)}\\&-m'(\xi)(C+E\widehat{C})\frac{e^{[\beta+\theta-m(\xi)]t_1}-\left[\beta+\theta-m\left(\xi\right)\right]t_1-1}{\left[\beta+\theta-m(\xi)\right]^2T}.\end{aligned}$$

Because $e^x x + x - 2e^x + 2 = \sum_{n=2}^{\infty} \frac{n-2}{n!} x^n > 0$ for all $x > 0$ and $0 < t_1 < T$, we then obtain

$$\begin{aligned}
g_1 >& \frac{m'(\xi)}{[\beta+\theta-m(\xi)]^2} \left\{ \beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C}) [\beta+\theta-m(\xi)] \right\} \\
&\times \left\{ e^{[\beta+\theta-m(\xi)]t_1} - 1 - e^{[\beta+\theta-m(\xi)]t_1} [\beta+\theta-m(\xi)] t_1 \right\} \\
&+ \frac{m'(\xi)}{[\beta+\theta-m(\xi)]^2} \left\{ \beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C}) [\beta+\theta-m(\xi)] \right\} \\
&\times \frac{e^{[\beta+\theta-m(\xi)]t_1} [\beta+\theta-m(\xi)] t_1 + [\beta+\theta-m(\xi)] t_1 - 2e^{[\beta+\theta-m(\xi)]t_1} + 2}{[\beta+\theta-m(\xi)] t_1} \\
=& \frac{m'(\xi)}{[\beta+\theta-m(\xi)]^3 t_1} \left\{ \beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C}) [\beta+\theta-m(\xi)] \right\} \\
&\times \left\{ 2e^{[\beta+\theta-m(\xi)]t_1} [\beta+\theta-m(\xi)] t_1 - e^{[\beta+\theta-m(\xi)]t_1} [\beta+\theta-m(\xi)]^2 t_1^2 \right. \\
&\left. +2 - 2e^{[\beta+\theta-m(\xi)]t_1} \right\} \\
=& -\frac{m'(\xi)}{[\beta+\theta-m(\xi)]^3 t_1} \left\{ \beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C}) [\beta+\theta-m(\xi)] \right\} \\
&\times \sum_{n=3}^{\infty} \frac{(n-2)(n-1)[\beta+\theta-m(\xi)]^n t_1^n}{(n+1)!} \\
>& 0
\end{aligned}$$

and

$$\begin{aligned}
g_2 >& m'(\xi)(C + E\widehat{C}) \left\{ \frac{e^{[\beta+\theta-m(\xi)]t_1} - 1}{\beta+\theta-m(\xi)} - \frac{e^{[\beta+\theta-m(\xi)]t_1} - [\beta+\theta-m(\xi)] t_1 - 1}{[\beta+\theta-m(\xi)]^2 t_1} \right\} \\
=& \frac{m'(\xi)(C + E\widehat{C})}{[\beta+\theta-m(\xi)]^2 t_1} \left\{ e^{[\beta+\theta-m(\xi)]t_1} [\beta+\theta-m(\xi)] t_1 - e^{[\beta+\theta-m(\xi)]t_1} + 1 \right\} \\
=& \frac{m'(\xi)(C + E\widehat{C})}{[\beta+\theta-m(\xi)]^2 t_1} \sum_{n=2}^{\infty} \frac{(n-1)[\beta+\theta-m(\xi)]^n t_1^n}{n!} \\
>& 0.
\end{aligned}$$

Combining the above inequalities, we observe that (A.1) holds if and only if $\frac{dt_1}{d\xi} > 0$. This together with $T > t_1 > 0$ and $\frac{dT}{d\xi} - \frac{dt_1}{d\xi} < 0$ yields the desired result that

$$\frac{d}{d\xi} \frac{t_1}{T} = \frac{1}{T^2} \left(T \frac{dt_1}{d\xi} - t_1 \frac{dT}{d\xi} \right) > \frac{1}{T^2} \left(t_1 \frac{dt_1}{d\xi} - t_1 \frac{dT}{d\xi} \right) = \frac{t_1}{T^2} \left(\frac{dt_1}{d\xi} - \frac{dT}{d\xi} \right) > 0.$$

Consequently, the optimal service level increases strictly in preservation technology investment, which completes the proof of the Proposition 1. □

Proof of Proposition 2 (1). Differentiating implicitly on both sides of (3.9) and (3.10) with respect to E and simplifying results in

$$\begin{aligned}
0 = & -\left\{\widehat{C}_1 + \widehat{C}\left[\beta+\theta-m(\xi)\right]\right\} \frac{e^{[\beta+\theta-m(\xi)]t_1}-1}{\beta+\theta-m(\xi)} - \widehat{C}\left[1-b(T-t_1)\right] \\
& + \left\{\beta S - (C_1 + E\widehat{C}_1) - (C+E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\} e^{[\beta+\theta-m(\xi)]t_1} \frac{dt_1}{dE} \\
& + \left\{C_2 b(T-t_1) - \left[S-(C+E\widehat{C})+C_3-C_2\left(T-t_1\right)\right]b'(T-t_1)\right\}\left(\frac{dT}{dE}-\frac{dt_1}{dE}\right) \\
= & \left\{\frac{CE(t_1,T,\xi)}{\alpha T} - \left\{\widehat{C}_1 + \widehat{C}\left[\beta+\theta-m(\xi)\right]\right\} \frac{e^{[\beta+\theta-m(\xi)]t_1}-1}{\beta+\theta-m(\xi)} - \widehat{C}\right\} \\
& + \left\{\beta S - (C_1 + E\widehat{C}_1) - (C+E\widehat{C})\left[\beta+\theta-m(\xi)\right]\right\} e^{[\beta+\theta-m(\xi)]t_1} \frac{dt_1}{dE} \qquad (A.3)
\end{aligned}$$

and

$$\begin{aligned}
\frac{CE(t_1,T,\xi)}{\alpha T} - \widehat{C}b\left(T-t_1\right) = & \left\{C_2 b(T-t_1) - \left[S-(C+E\widehat{C})+C_3-C_2(T-t_1)\right]b'(T-t_1)\right\} \\
& \times \left(\frac{dT}{dE}-\frac{dt_1}{dE}\right). \qquad (A.4)
\end{aligned}$$

Recall that $\beta S - (C_1 + E\widehat{C}_1) - (C+E\widehat{C})\left[\beta+\theta-m(\xi)\right] < 0$ and $S-(C+E\widehat{C})+C_3-C_2(T-t_1) > 0$, we can then observe from (A.3) and (A.4) that

$$\left\{\frac{CE(t_1,T,\xi)}{\alpha T} - \left\{\widehat{C}_1 + \widehat{C}\left[\beta+\theta-m(\xi)\right]\right\} \frac{e^{[\beta+\theta-m(\xi)]t_1}-1}{\beta+\theta-m(\xi)} - \widehat{C}\right\} \frac{dt_1}{dE} > 0 \qquad (A.5)$$

and

$$\left[\frac{CE(t_1,T,\xi)}{\alpha T} - \widehat{C}b\left(T-t_1\right)\right]\left(\frac{dT}{dE}-\frac{dt_1}{dE}\right) > 0. \qquad (A.6)$$

In order to prove $\frac{d}{dE}\frac{CE(t_1,T,\xi)}{T} < 0$, taking the first derivative of $\frac{CE(t_1,T,\xi)}{T}$ implicitly with respect to E and then simplifying leads to

$$\begin{aligned}
\frac{d}{dE}\frac{CE(t_1,T,\xi)}{T} = & \frac{\alpha}{T}\left\{\left\{\widehat{C}_1 + \widehat{C}\left[\beta+\theta-m(\xi)\right]\right\} \frac{e^{[\beta+\theta-m(\xi)]t_1}-1}{\beta+\theta-m(\xi)} + \widehat{C} - \frac{CE(t_1,T,\xi)}{\alpha T}\right\} \frac{dt_1}{dE} \\
& + \frac{\alpha}{T}\left[\widehat{C}b(T-t_1) - \frac{CE\left(t_1,T,\xi\right)}{\alpha T}\right]\left(\frac{dT}{dE}-\frac{dt_1}{dE}\right).
\end{aligned}$$

From the analysis carried out so far, it is straightforward to see that the proof is immediately evident from (A.5) and (A.6).

(2). By taking implicit differentiation on $\Pi(t_1,T,\xi)$ with respect to ϖ, we have

$$\frac{d\Pi(t_1,T,\xi)}{d\varpi} = \frac{\partial\Pi\left(t_1,T,\xi\right)}{\partial t_1}\frac{dt_1}{dE} + \frac{\partial\Pi(t_1,T,\xi)}{dT}\frac{dT}{dE} + \frac{\partial\Pi\left(t_1,T,\xi\right)}{\partial E}$$

$$= \varpi - \frac{CE(t_1, T, \xi)}{T}.$$

Using the fact that $\frac{d}{dE}\frac{CE(t_1,T,\xi)}{T} < 0$, it was relatively easy to show that $\frac{d\Pi(t_1,T,\xi)}{dE} > 0$ as $\frac{CE(t_1,T,\xi)}{T} \leq \varpi$ and $\frac{d\Pi(t_1,T,\xi)}{dE} \leq 0$ as $\frac{CE(t_1,T,\xi)}{T} > \varpi$, respectively. Therefore, the optimal total profit per unit time is strictly pseudoconvex in carbon price. This completes the proof. $\square$

Proof of Proposition 3. We begin with the first statement. Differentiating implicitly on both sides of Eqs. (3.9) and (3.10) with respect to ϖ simplifying gives

$$\begin{aligned}0 = &\left\{\beta S - (C_1 + E\widehat{C}_1) - (C + E\widehat{C})\left[\beta + \theta - m(\xi)\right]\right\} e^{[\beta+\theta-m(\xi)]t_1}\frac{dt_1}{d\varpi} \\ &+ \left\{C_2 b(T - t_1) - [S - (C + E\widehat{C}) + C_3 - C_2(T - t_1)]b'(T - t_1)\right\}\left(\frac{dT}{d\varpi} - \frac{dt_1}{d\varpi}\right)\end{aligned}$$

and

$$0 = \left\{C_2 b(T - t_1) - [S - (C + E\widehat{C}) + C_3 - C_2(T - t_1)]b'(T - t_1)\right\}\left(\frac{dT}{d\varpi} - \frac{dt_1}{d\varpi}\right).$$

Based on previous equations, the behavior of t_1 and T with regard to ϖ are $\frac{dt_1}{d\varpi} = 0$ and $\frac{dT}{d\varpi} = 0$, respectively. Therefore, it is clear to see that the carbon emissions per unit time remains constant as carbon cap rises.

Next, consider the relationship between $\Pi\,(t_1, T, \xi)$ and ϖ. By taking implicit differentiation on $\Pi(t_1, T, \xi)$ with respect to ϖ, we have

$$\frac{d\Pi(t_1, T, \xi)}{d\varpi} = \frac{\partial\Pi\,(t_1, T, \xi)}{\partial t_1}\frac{dt_1}{d\varpi} + \frac{\partial\Pi(t_1, T, \xi)}{\partial T}\frac{dT}{d\varpi} + \frac{\partial\Pi(t_1, T, \xi)}{\partial\varpi} = E.$$

That is, the optimal total profit per unit time increases linearly in carbon cap, which completes the proof of the Proposition 3. $\square$

Proof of Proposition 4 (1). The Karush-Kuhn-Tucker conditions for the Case are given as follows:

$$\frac{\partial}{\partial t_1}\frac{TP(t_1, T, \xi)}{T} - \lambda \times \frac{\partial}{\partial t_1}\frac{CE(t_1, T, \xi)}{T} = 0, \tag{A.7a}$$

$$\frac{\partial}{\partial T}\frac{TP(t_1, T, \xi)}{T} - \lambda \times \frac{\partial}{\partial T}\frac{CE(t_1, T, \xi)}{T} = 0, \tag{A.7b}$$

$$\frac{CE(t_1, T, \xi)}{T} - \varpi \leq 0, \tag{A.7c}$$

$$\lambda\left[\frac{CE(t_1, T, \xi)}{T} - \varpi\right] = 0, \tag{A.7d}$$

$$\lambda \geq 0, \tag{A.7e}$$

where λ is Lagrange multiplier. If $\lambda = 0$, the constrain $\frac{CE(t_1,T,\xi)}{T} - \varpi \leq 0$ is non-binding. Therefore, the uniqueness of global maximizer for the case follows immediately from

Theorem 1 by setting $E = 0$. On the other hand, if $\lambda > 0$, the problem attains its maximum at boundary, so $\frac{CE(t_1,T,\xi)}{T} = \varpi$. For a given λ, by using analogous arguments as in the proof of Theorem 1, we can show that the solution to Eqs. (A.7a) and (A.7b) is unique. Then, by a similar argument as in Proposition 2, we can show that $\frac{CE(t_1,T,\xi)}{T}$ is strictly decreasing in λ, and so there is a unique λ such that $\frac{CE(t_1,T,\xi)}{T} = \varpi$. Combining the above arguments together, we can conclude that the the global maximizer for Case 1 not only exists but is unique.

(2). Here we denote the corresponding carbon emissions per unit time in the model without carbon constrain as $\varpi^{\mathrm{wc}} = \frac{CE(t_1^{\mathrm{wc}},T^{\mathrm{wc}},\xi)}{T^{\mathrm{wc}}}$. If $\varpi \geq \varpi^{\mathrm{wc}}$, then the constrain $\frac{CE(t_1,T,\xi)}{T} - \varpi \leq 0$ is non-binding. Therefore, the optimal carbon emissions remains constant as carbon cap increases. On there other hand, if $\varpi < \varpi^{\mathrm{wc}}$, the optimal solution occurs at boundary, that is $\frac{CE(t_1,T,\xi)}{T} = \varpi$. Therefore, it is straightforward to see that $\frac{d}{d\varpi}\frac{CE(t_1,T,\xi)}{T} = 1$, as it implies that the optimal carbon emissions is increases linearly in carbon cap.

If $\varpi \geq \varpi^{\mathrm{wc}}$, the constrain $\frac{CE(t_1,T,\xi)}{T} - \varpi \leq 0$ is non-binding. Therefore, the optimal total profit per unit time remains constant as carbon cap increases. On there other hand, if $\varpi < \varpi^{\mathrm{wc}}$, we have

$$\begin{aligned}
\frac{d}{d\varpi}\frac{TP\left(t_1,T,\xi\right)}{T} &= \frac{\partial}{\partial t_1}\frac{TP\left(t_1,T,\xi\right)}{T}\frac{dt_1}{d\varpi} + \frac{\partial}{\partial T}\frac{TP\left(t_1,T,\xi\right)}{T}\frac{dT}{d\varpi} \\
&= \lambda\left\{\frac{\partial}{\partial t_1}\frac{CE(t_1,T,\xi)}{T}\frac{dt_1}{d\varpi} + \frac{\partial}{\partial T}\frac{CE(t_1,T,\xi)}{T}\frac{dT}{d\varpi}\right\} \\
&= \lambda > 0,
\end{aligned}$$

which implies that the optimal total profit per unit time increases strictly in carbon cap. This completes the proof of the Proposition 4. □

Proof of Proposition 5 (1). The Karush-Kuhn-Tucker conditions for the Case are given as follows:

$$\frac{\partial}{\partial t_1}\frac{TP(t_1,T,\xi)}{T} - (E-\lambda)\times\frac{\partial}{\partial t_1}\frac{CE(t_1,T,\xi)}{T} = 0, \tag{A.8a}$$

$$\frac{\partial}{\partial T}\frac{TP(t_1,T,\xi)}{T} - (E-\lambda)\times\frac{\partial}{\partial T}\frac{CE(t_1,T,\xi)}{T} = 0, \tag{A.8b}$$

$$\frac{CE(t_1,T,\xi)}{T} - \varpi > 0, \tag{A.8c}$$

$$\lambda\left[\frac{CE(t_1,T,\xi)}{T} - \varpi\right] = 0, \tag{A.8d}$$

$$\lambda \geq 0, \tag{A.8e}$$

where λ is Lagrange multiplier. If $\lambda = 0$, the constrain $\frac{CE(t_1,T,\xi)}{T} - \varpi > 0$ is non-binding. Hence, the uniqueness of global maximizer for the case follows immediately from Theorem 1. On the other hand, if $0 < \lambda \leq E$, the problem attains its maximum at boundary, so $\frac{CE(t_1,T,\xi)}{T} = \varpi$. For a given λ, by using analogous arguments as in the proof of Theorem 1, we can show that the solution to Eqs. (A.8a) and (A.8b) is unique. Then, by a similar argument as in Proposition 2, we can show that $\frac{CE(t_1,T,\xi)}{T}$ is strictly decreasing in $E - \lambda$, and so there is a unique λ such that $\frac{CE(t_1,T,\xi)}{T} = \varpi$. Combining the above arguments together, we can conclude that the the global maximizer for Case 2 not only exists but is unique.

(2). If $\varpi \le \varpi^{\mathrm{CT}}$, since the constrain $\frac{CE(t_1,T,\xi)}{T} - \varpi > 0$ is non-binding, the result that the optimal carbon emissions per unit time decreases strictly as carbon price increases follows immediately from the Proposition 2. Furthermore, since $\frac{CE(t_1,T,\xi)}{T}$ is strictly decreasing in E, there exists a unique E such that $\frac{CE(t_1,T,\xi)}{T} = \varpi$, say E^{L}. If $E > E^{\mathrm{L}}$, then $\frac{CE(t_1,T,\xi)}{T} < \varpi$, this contradicts the condition $\frac{CE(t_1,T,\xi)}{T} > \varpi$. Therefore, if $E > E^{\mathrm{L}}$, the optimal solution occurs at boundary, that is $\frac{CE(t_1,T,\xi)}{T} = \varpi$. Substituting this result into objective, we can find that the objective function is independent with E. And hence the optimal carbon emissions per unit time remains constant as carbon price increases. On the other hand, if $\varpi > \varpi^{\mathrm{CT}}$, since the optimal solution occurs at boundary, and therefore the optimal carbon emissions per unit time remains constant as carbon price increases.

(3). If $\varpi \le \varpi^{\mathrm{CT}}$, since the constrain $\frac{CE(t_1,T,\xi)}{T} - \varpi > 0$ is non-binding, the result that the optimal carbon emissions per unit time remains constant follows immediately from the Proposition 3. On the other hand, if $\varpi > \varpi^{\mathrm{CT}}$, the optimal solution occurs at boundary, that is $\frac{CE(t_1,T,\xi)}{T} = \varpi$. It is clear to see that $\frac{d}{d\varpi}\frac{CE(t_1,T,\xi)}{T} = 1$, as it implies that the optimal carbon emissions is increases linearly in carbon cap. Therefore, the optimal carbon emissions per unit time first remains constant, then increases linearly as carbon cap increases.

(4). If $\varpi \le \varpi^{\mathrm{CT}}$, it is trivial to see that the optimal carbon emissions per unit time is ϖ^{CT}. From Proposition 2, we obtain that

$$\frac{d}{dE}\left\{\frac{TP(t_1,T,\xi)}{T} - E\left[\frac{CE(t_1,T,\xi)}{T} - \varpi\right]\right\} = \varpi - \frac{CE(t_1,T,\xi)}{T} = \varpi - \varpi^{\mathrm{CT}} < 0.$$

And thus, the optimal total profit per unit time decreases strictly as carbon price increases. However, if $E > E^{\mathrm{L}}$, the optimal solution occurs at boundary, that is $\frac{CE(t_1,T,\xi)}{T} = \varpi$. By using same arguments as in (2), the optimal carbon emissions per unit time remains constant as carbon price increases. On the other hand, if $\varpi > \varpi^{\mathrm{CT}}$, the optimal solution occurs at boundary, that is $\frac{CE(t_1,T,\xi)}{T} = \varpi$. And therefore the optimal carbon emissions per unit time remains constant as carbon price increases.

(5). If $\varpi \le \varpi^{\mathrm{CT}}$, since the constrain $\frac{CE(t_1,T,\xi)}{T} - \varpi > 0$ is non-binding, the result that the optimal total profit per unit time increases linearly as carbon cap increases follows immediately from the Proposition 3. On the other hand, if $\varpi > \varpi^{\mathrm{CT}}$, the optimal solution occurs at boundary, that is $\frac{CE(t_1,T,\xi)}{T} = \varpi$. We then have

$$\frac{d}{d\varpi}\left\{\frac{TP(t_1,T,\xi)}{T} - E\left[\frac{CE(t_1,T,\xi)}{T} - \varpi\right]\right\}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial t_1}\frac{TP(t_1,T,\xi)}{T}\frac{dt_1}{d\varpi} - E\times\frac{\partial}{\partial t_1}\frac{CE(t_1,T,\xi)}{T}\frac{dT}{d\varpi} \\
&\quad + \frac{\partial}{\partial T}\frac{TP(t_1,T,\xi)}{T}\frac{dT}{d\varpi} - E\times\frac{\partial}{\partial T}\frac{CE(t_1,T,\xi)}{T}\frac{dT}{d\varpi} + E \\
&= -\lambda\left\{\frac{\partial}{\partial t_1}\frac{CE(t_1,T,\xi)}{T}\frac{dt_1}{d\varpi} + \frac{\partial}{\partial T}\frac{CE(t_1,T,\xi)}{T}\frac{dT}{d\varpi}\right\} + E \\
&= E - \lambda.
\end{aligned}
$$

Further, because $\frac{CE(t_1,T,\xi)}{T}$ is strictly decreasing in $E-\lambda$, λ is strictly increasing in ϖ. Denote $\lambda^*(\varpi)$ as the corresponding optimal λ with a given carbon cap ϖ. Then, there exists a unique ϖ such that $\lambda(\varpi)=E$, say ϖ^{E}. When $\varpi^{\mathrm{CE}} < \varpi \le \varpi^{\mathrm{E}}$, it implies that $0<\lambda^*(\varpi)\le E$. Since $E-\lambda^*(\varpi)>0$, the optimal total profit per unit time increases strictly as carbon cap increases. However, when $\varpi > \varpi^{\mathrm{E}}$, because $\lambda^*(\varpi)>E$, it is obvious to see that the optimal total profit per unit time decrease strictly as carbon cap increases. □

Proof of Proposition 6. From the discussion in Proposition 4, since $\frac{CE(t_1,T,\xi)}{T}$ decreases strictly in λ when $\varpi<\varpi^{\mathrm{WC}}$, λ is strictly decreasing in ϖ. So that λ can be written as a function of ϖ. Denote $\lambda^*(\varpi)$ as the corresponding optimal λ with a given carbon cap ϖ. Hence, we have $\lambda^*(\varpi^{\mathrm{WC}})=0$ and $\lambda^*(\varpi^{\mathrm{CT}})=E$. Moreover, because λ decreases strictly in ϖ, it straightforward to see that $\varpi^{\mathrm{CT}}<\varpi^{\mathrm{WC}}$.

First, if $\varpi \ge \varpi^{\mathrm{WC}}$, since the constrain $\frac{CE(t_1,T,\xi)}{T}-\varpi\le 0$ is non-binding, it is straightforward to see that the optimal strategy is Case 1. Next, when $\varpi^{\mathrm{CT}}<\varpi<\varpi^{\mathrm{WC}}$, the shadow price in Case 1 is $\lambda^*(\varpi)$ and $0<\lambda^*(\varpi)<E$. It clear to see that the shadow price in Case 1 is always less than its unit carbon cost, and thereby it is not profitable to buy extra emission rights. Therefore, when $\varpi^{\mathrm{CT}}<\varpi<\varpi^{\mathrm{WC}}$, the optimal strategy is Case 1. Finally, because $\lambda^*(\varpi)\ge E$ when $\varpi\le\varpi^{\mathrm{CT}}$, the shadow price in Case 1 is always greater or equal than its unit carbon cost and it is worthwhile to buy extra emission rights until $\varpi=\varpi^{\mathrm{CT}}$. Therefore, when $\varpi\le\varpi^{\mathrm{CT}}$, the optimal strategy is Case 2, which also completes the proof of the Proposition 6. □

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(Received March 2016; accepted September 2017)