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A New Approach in Multiple Attribute Decision Making Using Exponential Hesitant Fuzzy Entropy

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Abstract

Hesitant fuzzy sets (HFSs) proposed by Torra and Torra and Narukawa have been proved to be more practical in handling fuzziness. The feature of HFSs to assign membership degrees in the form of a set has made them very useful for solving multiple attribute decision making (MADM) problem. In this communication, we have introduced a new exponential hesitant fuzzy entropy based on well-known exponential entropy studied by Pal and Pal. The proposed entropy is then used for solving MADM problem. The attributes weights play an important role in solution of MADM problem. In this study, two methods are discussed to determine attributes weights. Finally, two numerical examples are given to illustrate the effectiveness and feasibility of proposed method.

Keywords: Hesitant fuzzy set, VIKOR, MADM.

1. Introduction

In a Multiple Attribute Decision Making (MADM) problem, our aim is to select the best alternative satisfying all the attributes. Sometimes, the criterion involved are so confusing and commensurate that to take a final call becomes a herculean task. Several methods and tools have been developed to solve MADM problems. Before the introduction of fuzzy set theory by Zadeh [52], probability was the only way to measure the uncertainty. But in day-to-day life, not all type of uncertainty can be quantified using probability, for example, low price, fast speed, very smart etc. This is due to the fact that such type of vague terms cannot be expressed as precise values. To handle such type of imprecise and imperfect information, fuzzy set theory proposed by Zadeh [52] has proved to be an effective tool. Judging the powerfulness of fuzzy set, many researchers from across the world attracted towards it and several extensions of fuzzy sets were proposed. Type-2 fuzzy sets (see Zadeh [53]), intuitionistic fuzzy sets (see Atanassov [1]), interval-valued intuitionistic fuzzy (see Atanassov and Gargove [2]) sets are some well-known extensions of fuzzy sets. The proposal of intuitionistic fuzzy entropy by Burillo and Bustince [4] caused the researchers across the world to introduce the information measures from their viewpoints (see Joshi and Kumar [15], Joshi and Kumar [20], Joshi and Kumar [21], Joshi [22], Joshi and Kumar [23]). All these extensions are based on the rationale that it is not clear to assign the membership degree of an element to a fixed set (see Torra and Narukawa [43], Torra [44]). All of the extensions of fuzzy sets including fuzzy sets themselves are based on one precise value of membership degree. But this is too ideal to match in our daily life. In practical, it is not always possible that all the decision makers assign same membership degree to an alternative with respect to same attribute. This can be better understood with the help of an example.

Consider an example of a company run by its governing council (GC) consisting of several decision makers. All the members of GC may have different qualifications, backgrounds, expertise and knowledge. If the company have to make a decision regarding an alternative corresponding to an attribute then all the members of GC may not assign the same membership degree to the alternative. Some members will provide .3 as assessment value, some will assign .6 and some others may give evaluation of .7. The three groups cannot persuade each other. Thus, the best way to completely represent this situation is the set of values given by $\{.3, .6, .7\}$ than crisp numbers or interval-valued fuzzy numbers, that is, $[.3, .7]$ or intuitionistic fuzzy numbers $(.3, .7)$. Thus, the above situation cannot be represented completely by using any of the extensions of fuzzy sets.

To handle such situation, Torra and Narukawa [38] and Torra [39] introduced a new generalization of fuzzy sets called Hesitant Fuzzy Sets (HFSs). Although, HFS does not allocate an exact membership degree, it depicts the fuzziness through a set of possible values of membership degree. Thus, HFSs are more closer to reality than other extensions of fuzzy sets. Due to its proximity to real world problems, many and many authors are attracting towards HFSs and giving their applications in distinct fields (see Rodriguez et al. [36], Wei [45], Xia and Xu [47], Xu and Xia [48], Liang et al. [28]). Recently, Singh and Lalotra [39], Singh et al. [40], Singh et al. [41] proposed generalized correlation coefficients and applied them to clustering analysis in hesitant fuzzy settings, Yang and Hussain [50] constructed new distance and similarity measures between HFSs based on Hausdorff metric and applied them in clustering, and Yao and Wang [51] proposed a new concept of hesitant intuitionistic fuzzy sets (HIFSs) to capture the uncertain information by refining dual HFS and gave their applications in medical diagnosis and decision making.

A MADM problem can be characterized by selecting the most suitable alternative from a set of feasible alternatives satisfying a certain set of attributes. Ratings of alternatives depend upon the criterion involved. The criterion for different problems are different. For example, the criteria for buying a car cannot be same as the criteria for selecting the school for the child. So, the criteria should be established according to the reality of the problem. A large amount of research has been done on evaluation of alternatives and several methods and techniques have been proposed so far in this regard. Ribeiro [37] surveyed all the methods in detail and made a comparison of all. However, there exists no solution satisfying the entire criterion simultaneously, this makes the decision making process very interactive. One common example is the relation between development possibility and protection of environment. Pareto [34] tried to address this problem by arguing that if one criteria is to be improved then some other will have to be made worst. This was the case when all alternatives are non-inferior, but this cannot be applied to the case where one best alternative is to be selected from a set of alternatives. The solution to this problem was provided by Yu [49] by suggesting a compromise solution using compromise programming. In this method, the alternative nearest to the ideal solution was considered as the best alternative. Based on the concept proposed by Yu [49], many decision making methods such as TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) (see Hwang and Yoon [12]), PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluations) (see Brans and Mareschal [3]) and VIKOR (Vlsekriterijumska Optimizacija i Kompromisno Resenje) (see Opricovic and Tzeng [31] etc. were introduced. On comparing the TOPSIS method and VIKOR method, Opricovic and Tzeng [31] pointed out that though the two methods are based on the relative distances from ideal solutions but the TOPSIS method does not provide the compromise solution. Furthermore, Opricovic and Tzeng [32] extended the VIKOR method by adding a stability analysis and established that use of VIKOR method is more advantageous than other existing methods. Recently, Liang et al. [28] extended the VIKOR method to pythagorean fuzzy environment and proposed two decision making methods by combining TODIM (see Gomes and Rangel [11]) and VIKOR methods.

From the above discussion, the importance of HFSs and VIKOR method can be easily judged. However, a lot of research has been done on solving MADM problems using fuzzy sets and intuitionistic fuzzy sets, but a very little research has been done on solving MADM problems where ratings of alternatives are expressed by using HFSs. This communication is a sequel in this direction. The prime aims of introducing this communication are: (1). To introduce an exponential hesitant fuzzy entropy based on the exponential entropy studied by Pal and Pal [33], (2). To introduce a new MADM method based on the proposed hesitant fuzzy entropy and using the concept of VIKOR method. To do so, the present communication is managed as follows: The contribution of earlier researchers in the field and the prime aim of this paper are given in the Section 1. Some basic concepts necessary to understand the topic under study are given in Section 2. A new hesitant fuzzy entropy is proposed in Section 3. A new MADM method based on the proposed entropy is introduced in Section 4. In Section 5, the proposed MADM method is explained with the help of two numerical examples. At last, the paper is concluded wih 'Conclusions' in Section 6.

In next Section, we give some basic concepts and definitions regarding HFSs.

2. Preliminaries

Definition 1 (Fuzzy Set (see Zadeh [52]). Let $X = (g_1, g_2, \ldots, g_n)$ be a finite universe of discourse. A fuzzy set $\widetilde{G}$ on X is defined as

$$\widetilde{G} = \{\langle g_i, \mu_{\widetilde{G}}(g_i)\rangle \mid g_i \in X\}, \tag{2.1}$$

where $\mu_{\widetilde{G}} : X \to [0, 1]$ represents the membership function and the number $\mu_{\widetilde{G}}(g_i)$ denotes the membership degree.

Torra and Narukawa [43] generalized the concept of FSs to HFSs as follows:

Definition 2 (Hesitant Fuzzy Set (see Torra and Narukawa [43], Torra [44]))**.** For a finite universe of discourse $X = (g_1, g_2, \ldots, g_n)$, the HFS is given by:

$$E = \{\langle g_i, \psi_E(g_i)\rangle \mid g_i \in X\}, \tag{2.2}$$

where $\psi_E : X \to [0,1]$ returns a subset of $[0,1]$ and $\psi_E(g_i)$ is a set of values in $[0,1]$. For convenience sake, Xia and Xu [42] named $\psi_E(g_i)$ as Hesitant Fuzzy Element (HFE) and set of all HFEs will be denoted by Ψ_E. In case, if there is only one value in $\psi_E(g_i)$ then it will be reduced to FS. In this way, HFS is the generalization of FS.

Definition 3 (Manhattan Distance)**.** For any two HFEs ψ_{E_1} and ψ_{E_2}, the Manhattan distance $d(\psi_{E_1}, \psi_{E_2})$ is given by

$$d(\psi_{E_1}, \psi_{E_2}) = \frac{1}{l_\psi} \sum_{i=1}^{l_\psi} \left| \psi_{E_1}^{\sigma(i)} - \psi_{E_2}^{\sigma(i)} \right|, \tag{2.3}$$

where $\psi_{E_1}^{\sigma(i)}$ and $\psi_{E_2}^{\sigma(i)}$ are the i^{th} largest values in ψ_{E_1} and ψ_{E_2}, respectively and l_ψ denotes the length of ψ_E. For example, consider two HFEs, that is, $\psi_{E_1}^{\sigma(i)} = (.6, .7, .8)$ and $\psi_{E_2} = (.4, .5, .6)$. Here, $l_\psi = 3$. Therefore, Manhattan distance between ψ_{E_1} and ψ_{E_2} is given by

$$d(\psi_{E_1}, \psi_{E_2}) = \frac{1}{l_\psi} \sum_{i=1}^{l_\psi} \left| \psi_{E_1}^{\sigma(i)} - \psi_{E_2}^{\sigma(i)} \right| = \frac{1}{3}(|.6 - .4| + |.7 - .5| + |.8 - .6|) = .2. \tag{2.4}$$

The above Example shows that the Manhattan distance of two HFEs is a crisp number. This suggests the way of defuzzification to convert HFEs into a crisp number which helps very much in deriving hesitant fuzzy VIKOR method.

From the literature review in Section 1, it can be easily observed that a very little research has been done about hesitant fuzzy information till now. Since entropy measures having wide application in areas like decision making (see Joshi and Kumar [14], Joshi and Kumar [16], Joshi and Kumar [17], Joshi and Kumar [18], Joshi and Kumar [19], Joshi and Kumar [20], Joshi and Kumar [21]), image processings (see Pal and King [35]), pattern recognition (see Li and Cheng [27]) therefore, it is necessary to develop some entropy measures under hesitant fuzzy environment. Now, we give axiomatic definition of hesitant fuzzy entropy as follows:

Definition 4 (Hesitant Fuzzy Entropy (see Xu and Xia [48]))**.** For an HFE ψ_E, a real valued function $\Theta : \Psi_E \to [0,1]$ is called hesitant fuzzy entropy if it satisfies the following properties

1. $\Theta(\psi_E) = 0$ if and only if $\psi_E = 0$ or $\psi_E = 1$.

2. $\Theta(\psi_E) = 1$ if and only if $\psi_{E_{\sigma(i)}} + \psi_{E_{\sigma(l_\psi - i+1)}} = 1$, for all $i = 1, 2, \ldots, l_\psi$ and l_ψ denotes the length of ψ_E.
3. For some $\psi_E, \psi_F \in \Psi_E$, $\Theta(\psi_E) \leq \Theta(\psi_F)$, if $\psi_{E_{\sigma(i)}} \leq \psi_{F_{\sigma(i)}}$ for $\psi_{E_{\sigma(i)}} + \psi_{F_{\sigma(i)}} \leq 1$ or $\psi_{E_{\sigma(i)}} \geq \psi_{F_{\sigma(i)}}$ for $\psi_{E_{\sigma(i)}} + \psi_{F_{\sigma(l_\psi - i+1)}} \geq 1$ for all $i = 1, 2, \ldots, l_\psi$.
4. $\Theta(\psi_E) = \Theta(\psi_E^c)$ where ψ_E^c denotes complement of ψ_E.

With these concepts in mind, in next Section, we introduce a new exponential hesitant fuzzy entropy.

3. A New Exponential Hesitant Fuzzy Entropy

3.1. Justification

Before its introduction, we justify the need of exponential entropy based on HFSs. The main benefit of using exponential function is its characteristic of possessing upper and lower bounds. In case, the decision maker assigns a very small weight to an attribute, that is, nearly equal to zero, then use of logarithmic entropy may give a very large value. In particular, if a decision maker assigns the weight-age to an attribute in the form of interval say $[0, 1]$, then as $x \to 0$, $\log(x) \to \infty$ and $\log(x)$ is not defined at $x = 0$. Also, as $x \to 1$, $\log(x) \to 0$ and at $x = 1$, $\log(x) = 0$ where 'x' denotes the probability of an event. In practice, the gain in information from an event, whether highly probable or highly unlikely, is expected to lie between two finite limits. This limits the scope of logarithmic entropy. Secondly, in Shannon entropy (see Shannon [42]), the measure of ignorance or gain in information is taken as $\log\left(\frac{1}{x}\right)$. But, statistically the ignorance can be better represented by $(1 - x)$ rather than $\frac{1}{x}$. If we define the gain in information $(\widetilde{\Delta} I(x))$ corresponding to an event with probability 'x' as $\widetilde{\Delta} I(x) = \log(1-x)$ or $\widetilde{\Delta} I(x) = -\log(1 - x)$, then either $\widetilde{\Delta} I < 0$ or approaches to infinity with increase in x which intuitively seems unappealing. The above problem can be circumvented by using exponential function. Moreover, as mentioned earlier, HFSs express the real life situations in a more general way than fuzzy sets, therefore, it becomes imperative to develop such measures and methods which can help in decision making in the problems involving exponential functions and HFSs. This study is a sequel in this direction.

3.2. Background

We start with probabilistic background. Let $\Delta_m = \{C = (c_1, c_2, \ldots, c_m) \mid c_i \geq 0, \sum_{i=1}^m c_i = 1\}$, $m \geq 2$, be a set of complete probability distributions. For some $C \in \Delta_m$, Pal and Pal [33] defined an exponential entropy given by

$$H_{pp}(C) = \sum_{i=1}^m c_i(e^{1-c_i} - 1), \tag{3.1}$$

The authors claimed many advantages of exponential entropy particularly in image processing. The concept of exponential entropy was generalized by Pal and Pal [33] to introduce a new entropy on fuzzy sets given by

$$\widetilde{H}_{pp}(C) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{m}[\mu_G(g_i)e^{(1-\mu_G(g_i))} + (1-\mu_G(g_i))e^{\mu_G(g_i)} - 1], \tag{3.2}$$

where $g_i \in X$. The measure (3.2) satisfies all the necessary requirements of fuzzy entropy proposed by De Luca and Termini [9].

In this paper, we further extend this idea from fuzzy sets to HFSs.

3.3. Definition

For any HFE ψ_E, we define a hesitant fuzzy entropy $H^{pp}(\psi_E)$ given by

$$H^{pp}(\psi_E) = \frac{1}{l_\psi(\sqrt{e}-1)}\sum_{i=1}^{m}\left[\left(\frac{\psi_{E_{\sigma(i)}} + \psi_{E_{\sigma(l_\psi-i+1)}}}{2}e^{\frac{2-\psi_{E_{\sigma(i)}}-\psi_{E_{\sigma(l_\psi-i+1)}}}{2}} + \frac{2-\psi_{E_{\sigma(i)}} - \psi_{E_{\sigma(l_\psi-i+1)}}}{2}e^{\frac{\psi_{E_{\sigma(i)}}+\psi_{E_{\sigma(l_\psi-i+1)}}}{2}}\right) - 1\right], \tag{3.3}$$

where l_ψ denotes the length of hesitant fuzzy element (HFE) ψ_E. Measure (3.3) satisfies all the properties mentioned in Definition 4.

4. Application of Proposed Entropy in Multiple Attribute Decision Making

In next Subsection, we introduce a new MADM method using the concept of VIKOR and based on proposed hesitant fuzzy entropy.

4.1. The proposed MADM method

In a MADM problem, sometimes the attributes are so complex and conflicting that to select the best alternative is a difficult task. The decision makers find it hard to express themselves as a single evaluation value. This may be due to lack of expertise about problem domain, time pressure etc. Therefore, in such situations, decision makers prefer to express themselves in the form of a set of values. Additionally, in some other situation, there may be a group of decision makers with different backgrounds. Due to this difference in backgrounds, it may not be possible to assign a consentaneous membership degrees to all alternatives corresponding to all attributes. This may cause a variation in evaluation of alternatives. Therefore, it is suitable to express the ratings of alternatives corresponding to different attributes using HFSs. Now, we present the VIKOR method to solve MADM problem with hesitant fuzzy information.

4.1.1. Determination of hesitant fuzzy decision matrix

Consider a MADM problem with $\Omega = (\Omega_i)_{i=1,2,\ldots,m}$ as a set of m-alternatives and $\chi = (\chi_j)_{j=1,2,\ldots,n}$ as a set of n-attributes. The HFS χ_j of i^{th} alternative on χ is given by $\chi_j = \{\langle \Omega_i, \psi_{\chi_j}(\Omega_i)\rangle \mid \Omega_i \in \Omega\}$, where $\psi_{\chi_j}(\Omega_i) = \{t \mid t \in \psi_{\chi_j}(\Omega_i), 0 \le t \le 1\}$ with

$i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Here, $\psi_{\chi_j}(\Omega_i)$ represents the set of possible values of membership degrees of the alternative Ω_i corresponding to attribute χ_j and will be denoted by ψ_{ij} in the rest of the paper. One important point that needs to be mentioned here is that all HFEs in a column must be of same length. For example, let ψ_{E_1} and ψ_{E_2} are two HFEs with respective lengths $l_{\psi_{E_1}}$ and $l_{\psi_{E_2}}$. If $l_{\psi_{E_1}} > l_{\psi_{E_2}}$, then length of ψ_{E_2} should be extended to length of ψ_{E_1} by adding the minimum value in it. In this paper, we have extended the shorter length by adding .5, that is, it is assumed that all the decision makers are compromise. For example, consider two HFEs $\psi_{E_1} = (.3, .4)$ and $\psi_{E_2} = (.1, .4, .6, .3)$. Now, length of ψ_{E_1} is 2 and length of ψ_{E_2} is 4. Therefore, we extend ψ_{E_1} as $(.3, .4, .5, .5)$. Thus, the hesitant fuzzy decision matrix H is given by

$$H = \begin{array}{c} \\ \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_m \end{array} \begin{array}{c} \begin{array}{cccc} \psi_{\chi_1} & \psi_{\chi_2} & \cdots & \psi_{\chi_n} \end{array} \\ \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{m1} & \psi_{m2} & \cdots & \psi_{mn} \end{bmatrix} \end{array}.$$

4.1.2. Determination of attributes weights

The attributes weights play an important role in the solution of a MADM problem. Since, each factor affecting the MADM problem has a different meaning, therefore, all the attributes cannot be assigned same weight. Realizing the importance of attributes weights, Chen and Li [5] bifurcated the methods of determining attributes weights into two catagories: Subjective methods and Objective methods. Subjective weights are according to the decision makers' preferences. AHP (Analytical Heirarchy Process) (see Saaty [38]), Delphi method (see Hwang and Lin [13]) and Weighted Least Square method (see Chu and Kalaba [6]) are the examples of subjective weights. Objective methods determine the attributes weights by solving mathematical models. Multi objective programming (see Choo and Wedely [7]), entropy method etc. belong to this category. However, the attributes weights assigned by the decision maker, if available, are of utmost importance, but due to paucity of time, or limited expertise about problem domain on the part of decision maker, or any other constraint, it is not always possible to have such reliable attributes weights. In case, when it is not possible to obtain such reliable subjective attributes weights, use of objective weights becomes useful. Therefore, determination of proper attributes weights is an important issue in the solution of MADM problems. Now, we discuss two cases to determine attributes weights. First is the case when attributes weights are completely unknown or incompletely known and second is the case, when we have partial information about them.

1. If Attributes Weights are Unknown:

If the information about attributes weights is incomplete or completely unknown, then attributes weights u_j can be calculated as follows (see Chen and Li [5]):

$$u_j = \frac{1 - \widetilde{E}_j}{n - \sum_{j=1}^{n} \widetilde{E}_j}, \qquad j = 1, 2, \ldots, n, \tag{4.1}$$

where $\widetilde{E}_j = \frac{1}{l_\psi} \sum_{i=1}^{l_\psi} H^{pp}(\psi_{ij})$, $j = 1, 2, \ldots, n$ and l_ψ denotes the length of ψ_E.

According to entropy theory, smaller value of entropy across alternatives provides decision makers a useful information. Therefore, criterion should be assigned a bigger weight; otherwise such a criterion will not be given due importance by most of the decision makers. In other words, such a criterion should be evaluated as a very small weight.

2. If Attributes Weights are Partially Known:

Sometimes, in real life, it is not possible for the decision makers to assign attributes weights as precised value. This may be due to lack of knowledge about problem domain, time pressure or lack of expertise on the part of decision makers. In such situations, decision makers prefer to express themselves in the form of intervals. Let the set of information provided by experts about attributes weights be denoted by $\hat{H}$. To compute the attributes weights in such a case, we use the method suggested by Wang and Wang [46] as follows:

The overall entropy $\overline{E}$ of an alternative Ω_i is given by

$$\overline{E}(\Omega_i) = \sum_{j=1}^{n} H^{pp}(\psi_{ij}). \tag{4.2}$$

Due to increasing complexities of real world problems, the decision makers may not be in position to provide exact attributes weights. Instead, they may possess only partial information about attributes weights (see Kim et al. [24]). To compute the attributes weights, we construct the following programming model:

$$\begin{aligned} \min E &= \sum_{i=1}^{l_\psi} u_j \overline{E}(\Omega_i) = \sum_{i=1}^{l_\psi} u_j \Big(\sum_{j=1}^{n} H^{pp}(\psi_{ij}) \Big) \\ &= \sum_{j=1}^{n} \frac{1}{\sqrt{e} - 1} \sum_{i=1}^{l_\psi} \frac{1}{l_\psi} \Bigg[\Bigg(\frac{\psi_{E_{\sigma(i)}} + \psi_{E_{\sigma(l_\psi - i+1)}}}{2} e^{\frac{2 - \psi_{E_{\sigma(i)}} - \psi_{E_{\sigma(l_\psi - i+1)}}}{2}} \\ &\qquad + \frac{2 - \psi_{E_{\sigma(i)}} - \psi_{E_{\sigma(l_\psi - i+1)}}}{2} e^{\frac{\psi_{E_{\sigma(i)}} + \psi_{E_{\sigma(l_\psi - i+1)}}}{2}} \Bigg) - 1 \Bigg], \end{aligned} \tag{4.3}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ is attributes weight vector such that $\sum_{j=1}^{n} u_j = 1$. Hence, by solving (4.3), the optimal solution $\boldsymbol{u} = \arg\min E$ is choosen as optimal attribute weights.

4.1.3. Determination of Ideal Solutions

Since the core idea of VIKOR method is based on the distance measures of alternatives from ideal solutions, therefore, determination of ideal solutions is necessary to rank the alternatives. For this, consider a HFE $\psi_E = (\mu_1, \mu_2, \ldots, \mu_t)$ where μ_k $(1 \leq k \leq t)$ represents the membership degrees assigned by decision makers to alternative Ω_i corresponding to attribute Λ_j. We define positive ideal solution ψ_{E^*} as follows:

1. For benifit criteria

$$\psi_{E^*} = \sup_{1 \leq k \leq t} (\mu_k) = (\underbrace{1, 1, \ldots, 1}_{t \text{ times}}). \tag{4.4}$$

2. For cost criteria

$$\psi_{E^*} = \inf_{1 \leq k \leq t} (\mu_k) = (\underbrace{0, 0, \ldots, 0}_{t \text{ times}}). \tag{4.5}$$

Similarly, we may define negative ideal solution ψ_{E_*} given by

1. For benifit criteria

$$\psi_{E_*} = \inf_{1 \leq k \leq t} (\mu_k) = (\underbrace{0, 0, \ldots, 0}_{t \text{ times}}). \tag{4.6}$$

2. For cost criteria

$$\psi_{E_*} = \sup_{1 \leq k \leq t} (\mu_k) = (\underbrace{1, 1, \ldots, 1}_{t \text{ times}}). \tag{4.7}$$

4.1.4. Compromise Solution

The VIKOR method (see Opricovic [29]) is based on the measure of closeness to the ideal solution. It is an efficient tool which provides the compromise solution from a set of conflicting criteria. The basic measure of compromise solution has originated from L^p-metric (see Yu [49]). For an MADM problem given in Subsection 4.1.1, Yu [49] proposed a compromise programming given by:

$$L_{p,i} = \left[\sum_{j=1}^{n} \left(u_j \frac{\psi_{E^*} - \psi_{ij}}{\psi_{E^*} - \psi_{E_*}} \right)^p \right]^{\frac{1}{p}}, \quad 1 \leq p \leq \infty, \ i = 1, 2, \ldots, m. \tag{4.8}$$

where ψ_{E^*} and ψ_{E_*} as defined in Subsection 4.1.3 respectively denote the positive and negative ideal solutions and u_j $(j = 1, 2, \ldots, n)$ represent the corresponding weight of j^{th} criteria.

Since it is not possible to satisfy all the criterion simultaneously, therefore, ideal solutions are infeasible in majority of the cases. Therefore, we try to find out a solution which is close to ideal solution, that is, compromise solution. VIKOR method provides compromise solution by integrating maximum group utility $(L_{1,i})$ and minimum individual regret $(L_{\infty,i})$. But, practically it is difficult to compute $(L_{1,i})$ and $(L_{\infty,i})$. Therefore, we use Manhattan distance to rewrite (4.8) as follows:

$$\hat{L}_{p,i} = \left[\sum_{j=1}^{n} \left(u_j \frac{d(\psi_{E^*}, \psi_{ij})}{d(\psi_{E^*}, \psi_{E_*})} \right)^p \right]^{\frac{1}{p}}, \quad 1 \leq p \leq \infty,\ i = 1, 2, \ldots, m. \tag{4.9}$$

where u_j $(j = 1, 2, \ldots, n)$ denotes the weight of j^{th} criteria satisfying $0 \leq u_j \leq 1$ and $\sum_{j=1}^{n} u_j = 1$. $d(\psi_{E^*}, \psi_{ij})$ and $d(\psi_{E^*}, \psi_{E_*})$ can be computed by using Manhattan distance given by Definition 3 Till now, we have defined the Manhattan L^p-metric for benifit type criteria. Similarly, it may be defined for cost criteria. Based on (4.9), we define hesitant fuzzy group utility $(\hat{S}_i)$ and hesitant fuzzy individual regret $(\hat{R}_i)$ for benifit type criterian as follows:

$$\hat{S}_i = \hat{L}_{1,i} = \sum_{j=1}^{n} u_j \frac{d(\psi_{E^*}, \psi_{ij})}{d(\psi_{E^*}, \psi_{E_*})} \tag{4.10}$$

and

$$\hat{R}_i = \hat{L}_{\infty,i} = \max_j \left(u_j \frac{d(\psi_{E^*}, \psi_{ij})}{d(\psi_{E^*}, \psi_{E_*})} \right) \tag{4.11}$$

where u_j $(j = 1, 2, \ldots, n)$ denotes the weight of j^{th} criteria satisfying $0 \leq u_j \leq 1$ and $\sum_{j=1}^{n} u_j = 1$. Distances $d(\psi_{E^*}, \psi_{ij}), d(\psi_{E^*}, \psi_{E_*})$ can be computed by using Definition 3. The hesitant fuzzy compromise measure $(\hat{Q}_i)$ corresponding to (4.9) is defined by:

$$\hat{Q}_i = \mu \frac{\hat{S}_i - \hat{S}_*}{\hat{S}^* - \hat{S}_*} + (1 - \mu) \frac{\hat{R}_i - \hat{R}_*}{\hat{R}^* - \hat{R}_*} \tag{4.12}$$

where $\hat{S}^* = \max_i \hat{S}_i$, $\hat{S}_* = \min_i \hat{S}_i$, $\hat{R}^* = \max_i \hat{R}_i$, $\hat{R}_* = \min_i \hat{R}_i$ and μ denotes the weight of the strategy of the majority of criteria or maximum overall utility. From (4.12), it is clear that hesitant fuzzy compromise solution combines maximum group utility and minimum individual regret.

4.1.5. Procedural steps of proposed MADM method

1. Construct the hesitant fuzzy decision matrix. The entries in matrix are the assessments of decision makers of different alternatives for different criterion. The weight u_j of j^{th} criteria is also decided by the decision makers.
2. Compute positive ideal solution (ψ_{E^*}) and negative ideal solution (ψ_{E_*}) for benifit criterion as well as cost criterion using (4.4) to (4.7).
3. Determine the weights of the criterion involved by using (4.1) and (4.3).
4. Compute the values of maximum group utility $(\hat{S}_i)$, minimum individual regret $(\hat{R}_i)$ and compromise solution $(\hat{Q}_i)$ by using (4.10), (4.11) and (4.12), respectively.

5. Rank the alternatives according to the values of $\hat{S}_i$, $\hat{R}_i$ and $\hat{Q}_i$ $(i = 1, 2, \ldots, m)$. The most suitable solution must satisfy the following two conditions:

C1 (Acceptable Advantage): If $\hat{Q}(\Omega^2) - \hat{Q}(\Omega^1) \geq \dfrac{1}{j-1}$ where Ω^1 and Ω^2 are the alternatives standing at first and second positions, respectively.

C2 (Acceptable Stability): Alternative Ω^1 should also be ranked at first position by $\hat{S}_i$ and $\hat{R}_i$.

The compromise solution is stable within a decision making process, which could be: "voting by majority rule" (when $\mu > 0.5$ is needed) or "by consensus" $\mu \approx 0.5$, or "with veto" $(\mu < 0.5)$. Here, μ is the weight of decision making strategy "the majority of criteria" (or "maximum group utility").

If these two conditions are not satisfied simultaneously, then we seek for compromise solution as follows:

(a) If condition **C1** is not satisfied, the find the maximum value of N for which

$$\hat{Q}(\Omega^N) - \hat{Q}(\Omega^1) < \frac{1}{j-1}; \tag{4.13}$$

then Ω^i $(i = 1, 2, \ldots, N)$ are the compromise solutions.

(b) If the condition **C2** is not satisfied, the Ω^1 and Ω^2 are the compromise solutions.

To understand the above procedure more clearly, we summarize it through flow chart as follows:

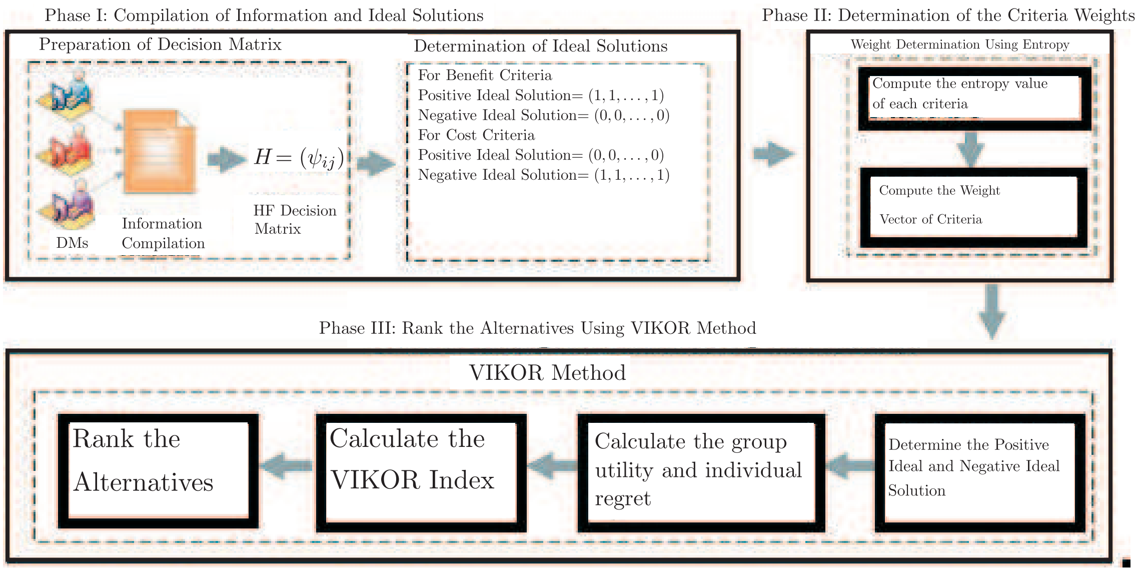

Figure 1: Graphical Abstract of Proposed MCDM Method.

In next Section, we consider a numerical Example to explain the above method.

5. Numerical Examples

5.1. When attributes weights are unknown

Consider a construction company ABC Ltd. who wants some potential suppliers to procure the raw material required for construction. Bids are invited for the purpose and company has received 4 tenders say $\Omega_1, \Omega_2, \Omega_3, \Omega_4$. Now, the company wants to enlist the four suppliers as a sequence of preferences. Four criterion namely (1). Quality (Λ_1), (2). Proximity to site (Λ_2), (3). Responsiveness (Λ_3), (4). Cost (Λ_4) are fixed to select the most appropriate supplier. To ensure a fair selection of the supplier, company has decided to hire the services of experts with different backgrounds and expertise. Depending upon their expertise and background, the membership degrees assigned by experts are compiled in Table 1. For example, to compute the degrees that the alternative Ω_i satisfies Λ_j, some decision makers assign .6, some assign .7 and some others assign .9, and these three groups cannot persuade each other, therefore, the dgerees that the alternative Ω_i satisfies Λ_j should be considered as HFE $\{.6, .7, .9\}$.

Table 1: Responses of Decision Makers.

	Λ_1	Λ_2	Λ_3	Λ_4
Ω_1	(.6, .7, .9)	(.6, .8)	(.3, .6, .9)	(.4, .5, .9)
Ω_2	(.7, .8, .9)	(.5, .8, .9)	(.4, .8)	(.5, .6, .7)
Ω_3	(.5, .6, .8)	(.6, .7, .9)	(.3, .5, .7)	(.5, .7)
Ω_4	(.6, .9)	(.7, .9)	(.2, .4, .7)	(.4, .5)

Since all the HFEs in Table 1 are not of same length, therefore, we extend the lengths of shorter HFEs by adding .5 in it so that all HFEs in each column have the same length and arrange them in ascending order. Thus, the hesitant fuzzy decision matrix so obtained is shown in Table 2.

Table 2: Hesitant Fuzzy Decision Matrix.

	Λ_1	Λ_2	Λ_3	Λ_4
Ω_1	(.6, .7, .9)	(.5, .6, .8)	(.3, .6, .9)	(.4, .5, .9)
Ω_2	(.7, .8, .9)	(.5, .8, .9)	(.4, .5, .8)	(.5, .6, .7)
Ω_3	(.5, .6, .8)	(.6, .7, .9)	(.3, .5, .7)	(.5, .5, .7)
Ω_4	(.5, .6, .9)	(.5, .7, .9)	(.2, .4, .7)	(.4, .5, .5)

Now, we compute the attributes weights. To assign the weight-age to different attributes, it is necessary to compute the information obtained from each attribute across all the alternatives. The more will be the information obtained from an attribute, the more will be weight-age assigned to it in decision making. For this, we compute information matrix using (3.3) given by Table 3.

Table 3: Hesitant Fuzzy Information Matrix.

	Λ_1	Λ_2	Λ_3	Λ_4
Ω_1	.7880	.9298	.9618	.9426
Ω_2	.6509	.7812	.9745	.9618
Ω_3	.9298	.7880	1	.9745
Ω_4	.8848	.8463	.9809	.9936
$\widetilde{E}_j$	3.2535	3.3453	3.9173	3.8725

The computed values of attributes weights using (4.1), when they are unknown to us are given by

$$u_1 = .2169, \quad u_2 = .2258, \quad u_2 = .2808, \quad u_4 = .2765. \tag{5.1}$$

Since, the traditional VIKOR method is based on the particular distance measure of closeness to the ideal solutions, in order to apply the HF-VIKOR method, we need to find out the ideal solutions. The computed values of ψ_{E^*} and ψ_{E_*} using (4.4) to (4.7) are shown in Table 4.

Table 4: Computed Values of ψ_{E^*} and ψ_{E_*}.

	Λ_1	Λ_2	Λ_3	Λ_4
$\psi_{E_j^*}$	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(0, 0, 0)
$\psi_{E_{j*}}$	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(1, 1, 1)

Now, we compute the values of $\hat{S}_i$, $\hat{R}_i$ and $\hat{Q}_i$ using (4.10), (4.11) and (4.12). Here, $\hat{S}_i$ denotes the hesitant fuzzy group utility as it fuses the normalized distances between ideal and actual evaluations determined by decision makers corresponding to each attribute. $\hat{R}_i$ denotes the hesitant fuzzy individual regret by using the weighted Manhattan distances between ideal and actual evaluation values determined by decision makers. The hesitant fuzzy compromise measure $\hat{Q}_i$ is the summation of the normalized hesitant fuzzy group utility values and the normalized hesitant fuzzy individual regret values. The computed values of $\hat{S}_i$, $\hat{R}_i$ and $\hat{Q}_i$ are given in Table 5. The ranking of alternatives Ω_1, Ω_2, Ω_3 and Ω_4 is given in Table 6.

Table 5. Computed Values of $\hat{S}_i$, $\hat{R}_i$ and $\hat{Q}_i$.

	Ω_1	Ω_2	Ω_3	Ω_4
$\hat{S}_i$	.4189	.3912	.4368	.4282
$\hat{R}_i$	.1659	.1659	.1567	.1591
$\hat{Q}_i$	.8037	.5000	.5000	.5361

Table 6. Ranks of Alternatives.

	Ω_1	Ω_2	Ω_3	Ω_4
$\hat{S}_i$	II	I	IV	III
$\hat{R}_i$	II	II	I	IV
$\hat{Q}_i$	III	I	I	II

Thus, the preferential sequences of alternatives are given by

$$\begin{aligned} &\text{By } \hat{Q}_i, && \Omega_2 = \Omega_3 \succ \Omega_4 \succ \Omega_1; \\ &\text{By } \hat{R}_i, && \Omega_3 \succ \Omega_2 = \Omega_1 \succ \Omega_4; \\ &\text{By } \hat{S}_i, && \Omega_2 \succ \Omega_1 \succ \Omega_4 \succ \Omega_3. \end{aligned} \tag{5.2}$$

Now,

$$\hat{Q}(\Omega_2) - \hat{Q}(\Omega_3) = 0 < \frac{1}{4-1} = .333, \tag{5.3}$$

therefore, condition C1 is not satisfied. Also, $\hat{Q}(\Omega_4) - \hat{Q}(\Omega_2) = .0361 < \frac{1}{4-1} = .333$, by applying condition C1, we have $\Omega_1, \Omega_2, \Omega_3$ as the compromise solution. Again, Ω_2 is not ranked best by $\hat{R}_i$, therefore, condition C2 ia also not satisfied. Thus, Ω_2, Ω_3 is the compromise solution.

A Comparative Analysis: To check the effectiveness of the proposed method, same example was computed by using VIKOR method for HFSs introduced by Liao and Xu [25]. As a result, we obtain Ω_2 as the best alternative. Therefore, it is quite natural to ask: which output is more reliable? It is a fact that each algorithm is defined with a different viewpoint. As discussed earlier, the attributes weights play a decisive role in the solution of a MADM problem. Due to the difference in viewpoints, the authors may assign different weight-age to different attributes. In VIKOR method suggested by Liao and Xu [25], the weights assigned to attributes are subjective weights whereas in the proposed method, we use objective attributes weights. Subjective weights are assigned according to preference decision makers. In practice, due to limited expertise about problem domain or shortage of time etc. that decision makers may not provide the justified attributes weights. When it becomes difficult to obtain such reliable attributes weights, the use of objective method becomes helpful. Thus, the output of proposed method is more reliable.

5.2. When attributes weights are partially known

Due to time constraints, lack of knowledge about problem domain etc., it may not be possible every time for decision makers to provide the attributes weights as fixed real numbers. In such cases, decision makers prefer to express themselves in the form of intervals. In view of this, we solve above Example for the case when we have partial information about attributes weights. Let the partial information available about attributes weights is denoted by $\hat{H}$ given by

$$\hat{H} = \{0 \leq u_1 \leq .25,\ 1 \leq u_2 \leq .15,\ .25 \leq u_3 \leq .35,\ .2 \leq u_4 \leq .45\}. \tag{5.4}$$

The computed hesitant fuzzy information matrix from the hesitant fuzzy decision matrix given by Table 2 is shown in Table 3. To determine the attributes weights, we construct the following programming model by using (4.3):

$$\min E = 3.2535u_1 + 3.3453u_2 + 3.9173u_3 + 3.8725u_4$$

$$\text{subject to} \begin{cases} 0 \leq u_1 \leq .25, \\ 1 \leq u_2 \leq .15, \\ .25 \leq u_3 \leq .35, \\ .2 \leq u_4 \leq .45 \\ u_1 + u_2 + u_3 + u_4 = 1. \end{cases} \tag{5.5}$$

Computing (5.5) using MATLAB software, the attributes weight vector so obtained is given by

$$U = (.25, .15, .25, .35)^T. \tag{5.6}$$

Now, we determine the values of $\hat{S}_i$, $\hat{R}_i$ and $\hat{Q}_i$ using (4.10), (4.11) and (4.12). The computed values of $\hat{S}_i$, $\hat{R}_i$ and $\hat{Q}_i$ are shown in Table 7 and ranks of alternatives are given in Table 8.

Table 7. Computed Values of $\hat{S}_i$, $\hat{R}_i$ and $\hat{Q}_i$.

	Ω_1	Ω_2	Ω_3	Ω_4
$\hat{S}_i$	.4317	.4083	.4550	.4333
$\hat{R}_i$	.2100	.2100	.1983	.1633
$\hat{Q}_i$	.7505	.5000	.8747	.2677

Table 8. Ranks of Alternatives.

	Ω_1	Ω_2	Ω_3	Ω_4
$\hat{S}_i$	II	I	IV	III
$\hat{R}_i$	III	III	II	I
$\hat{Q}_i$	III	II	IV	I

The ranks of alternatives Ω_i's are as follows:

$$\begin{array}{ll} \text{By } \hat{S}_i, & \Omega_2 \succ \Omega_1 \succ \Omega_4 \succ \Omega_3. \\ \text{By } \hat{R}_i, & \Omega_4 \succ \Omega_3 \succ \Omega_2 = \Omega_1; \\ \text{By } \hat{Q}_i, & \Omega_4 \succ \Omega_2 \succ \Omega_1 \succ \Omega_3; \end{array} \tag{5.7}$$

Thus from (5.7), it is clear that Ω_4 and Ω_2 stands at first and second place in the list of $\hat{Q}_i$ and $\hat{Q}(\Omega_2) - \hat{Q}(\Omega_4) = .2323 < \frac{1}{4-1} = .3333$. This shows that condition C1 is not satisfied. Also, for no other value of $\hat{Q}_i$ except Ω_2 and Ω_4, (4.13) holds. This implies that Ω_4, Ω_2 is the compromise solution. Again, Ω_4 which is ranked first by $\hat{Q}_i$ is not the best alternative in the list of $\hat{S}_i$. Therefore, condition C2 is also not satisfied. Thus, we have Ω_4, Ω_2 as compromise solution.

6 Conclusions

Hesitant fuzzy sets proposed by Torra [44] and Torra and Narukawa [43] have been proved to be more practical to depict the practical situation. In this communication, we have successfully introduced an exponential hesitant fuzzy entropy based on the HFSs and exponential entropy studied by Pal and Pal [33]. Attributes weights play an important role in the solution of MADM problem. Two methods of finding attributes weights are discussed. First method deals with the case of completely unknown attributes weights and second method describes the case of partially known attributes weights. The main limitation of this study is that it is based on complete probability distribution, that is, for a probability distribution $(c_1, c_2, \ldots, c_n)$ for which $\sum_{i=1}^{n} c_i = 1$ holds which may not always be the case, for example, D-numbers proposed by Deng [10] for which $\sum_{i=1}^{n} c_i \leq 1$. Another limitation of the proposed study lies with type of data used. The proposed work is based on HFSs where decision makers are expected to express their viewpoints as precise numbers. In practical, due to limited expertise about problem domain or due to shortage of time etc. that decision makers prefer to express themselves in the form of intervals instead of precise numbers. To cover up all such cases, the proposed measure as well as method may further be extended to include more general cases like interval-valued HFSs. The more general work is under consideration and will be reported somewhere else.

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