Decision making with incomplete information – Some new developments

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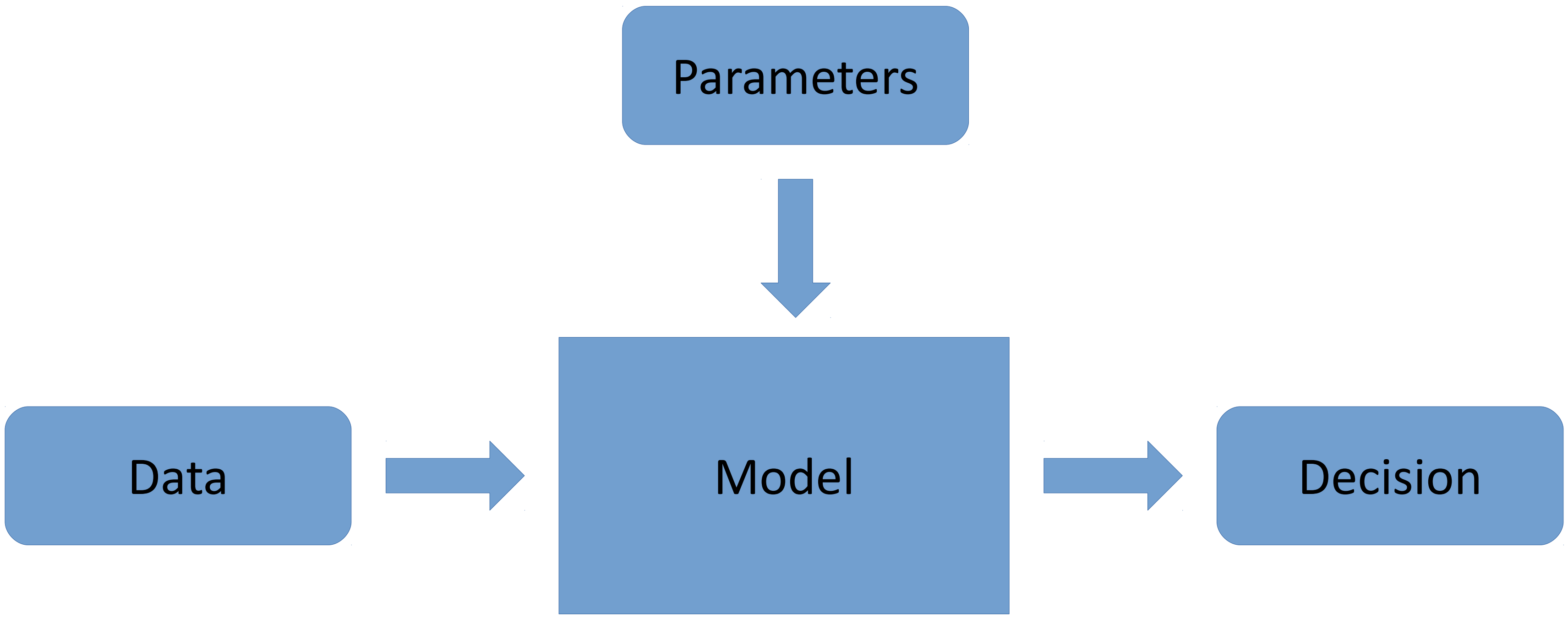
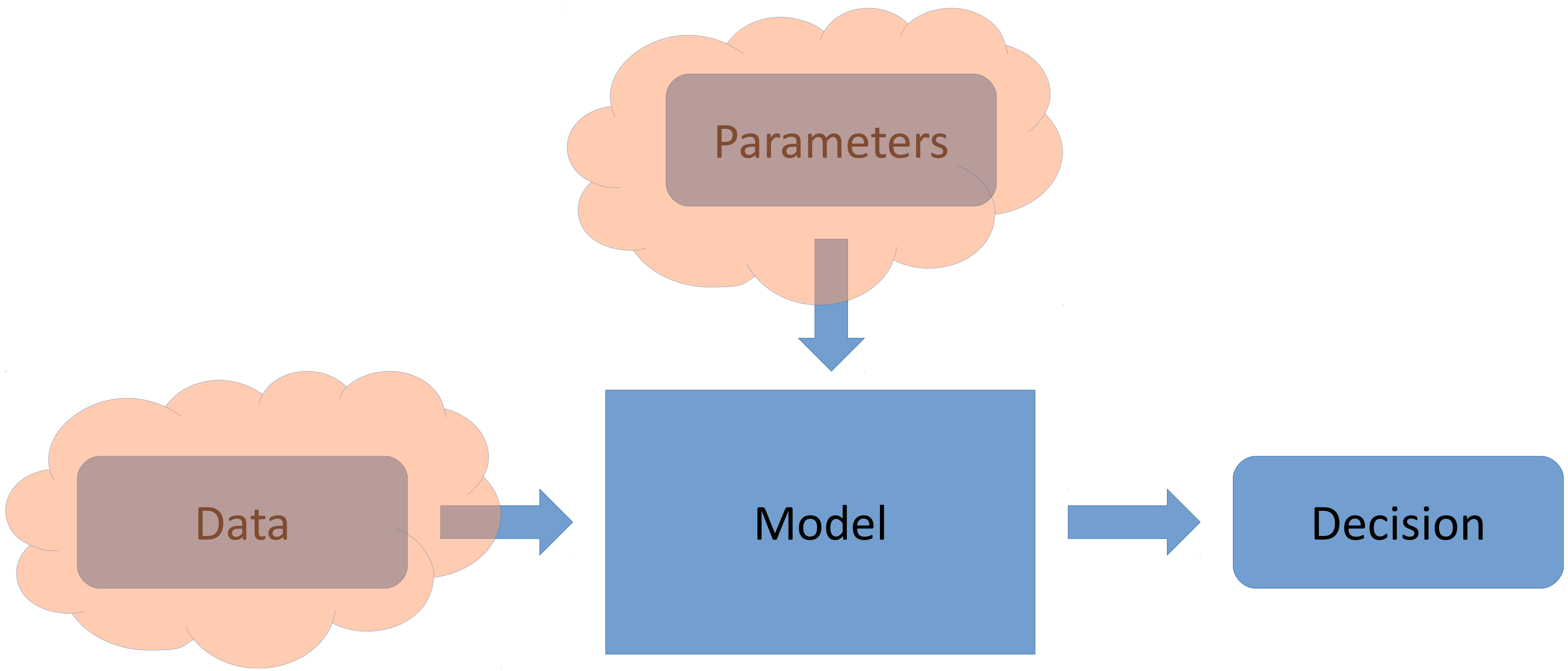
Agenda

- Problem description
- Overview of methods
- Single parameter approaches
- Relation based approaches
- Volume based (probabilistic) approaches
- A new concept: rankings from probabilistic statements
- Conclusions

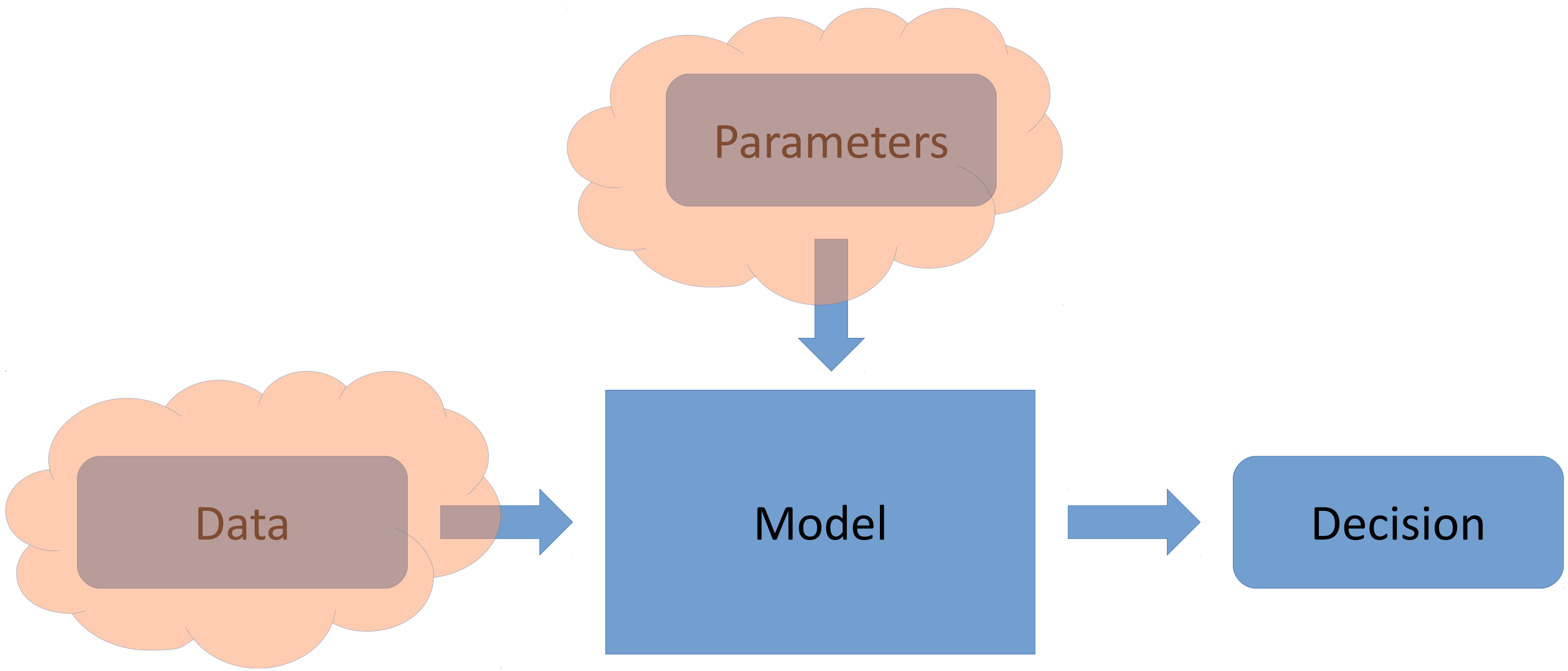
Problem description



Problem description



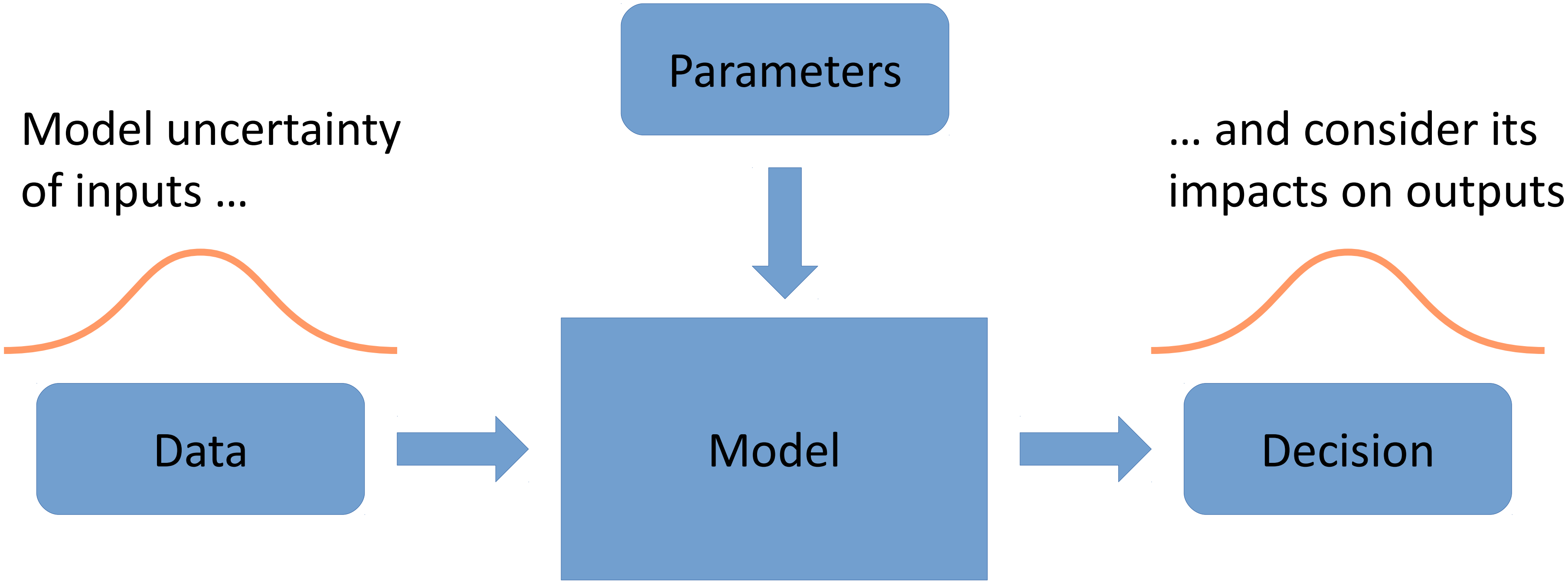
Problem description



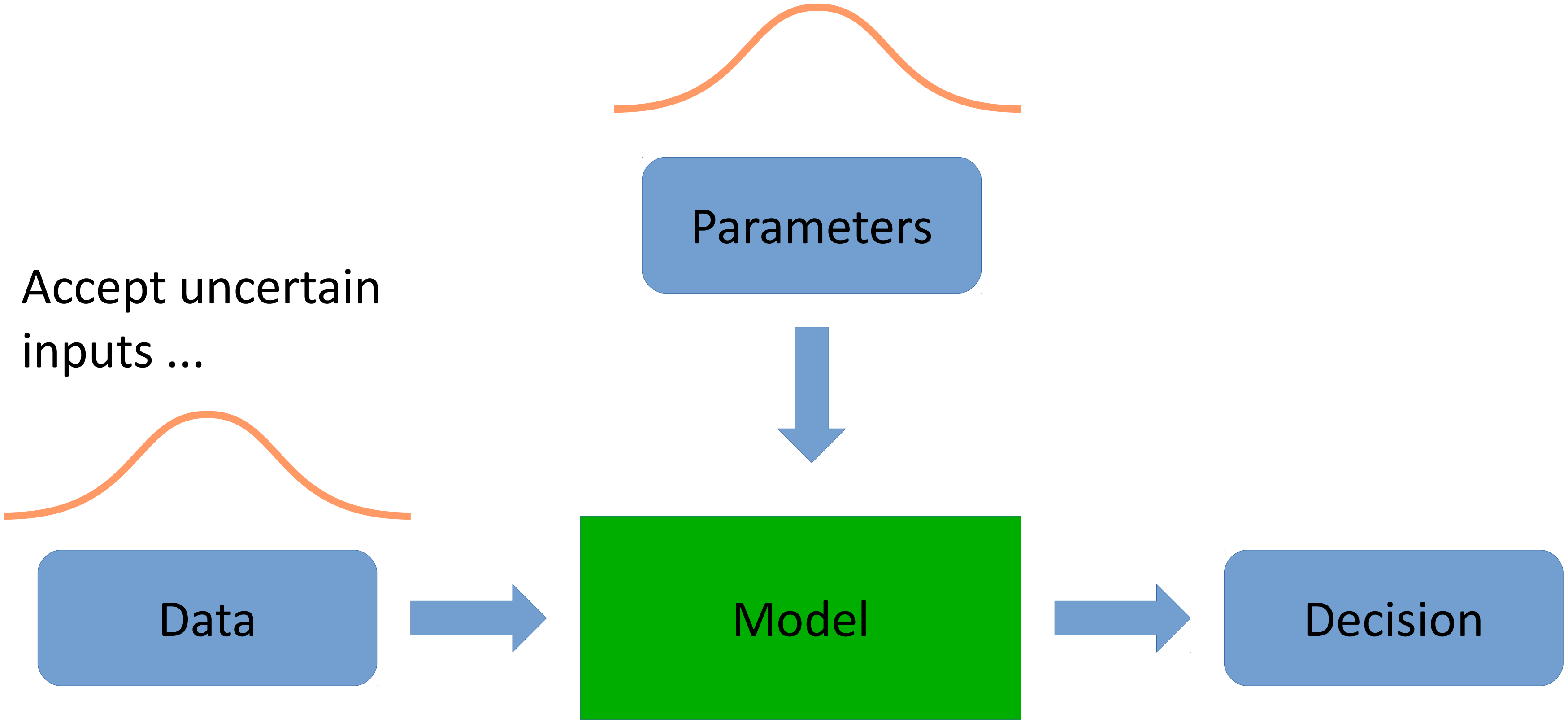
How to deal with parameters and input data
which are not known with certainty?

Sensitivity analysis

The usual approach: Sensitivity analysis



Decision making with incomplete information



… account for uncertainty in the model …

… and obtain a decision that takes into account uncertainty

Example area of application

universität wien

Additive multi-attribute utility with unknown weights:

$$u(X) = \sum_k w_k u_k(x_k)$$

Note: most concepts can also be applied to

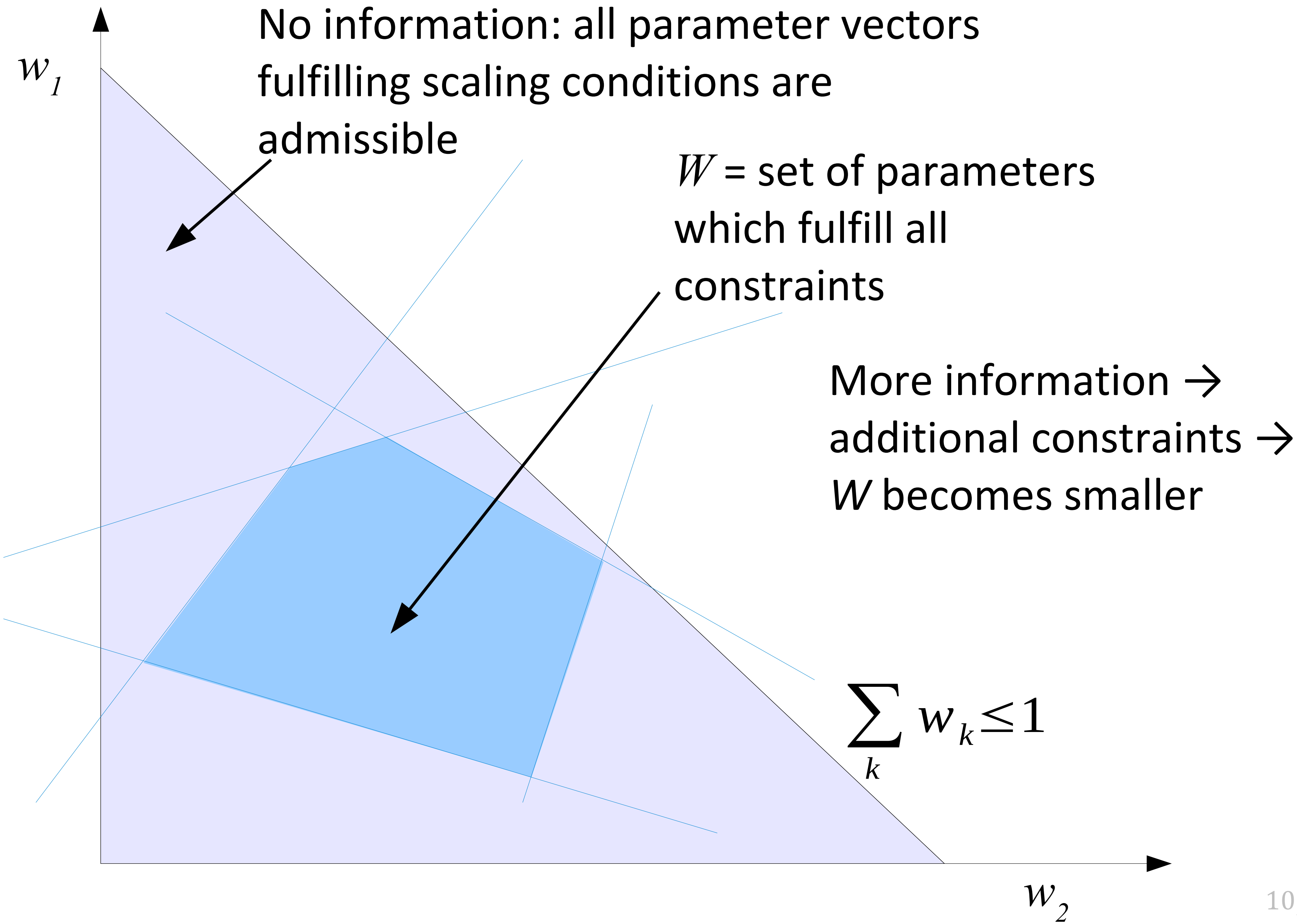
- **other uncertain parameters**
  (e.g. partial value functions)
- **other preference models**
  (e.g. outranking models)
- **other domains**
  (e.g. risk, group decisions, ...)

Forms of incomplete information

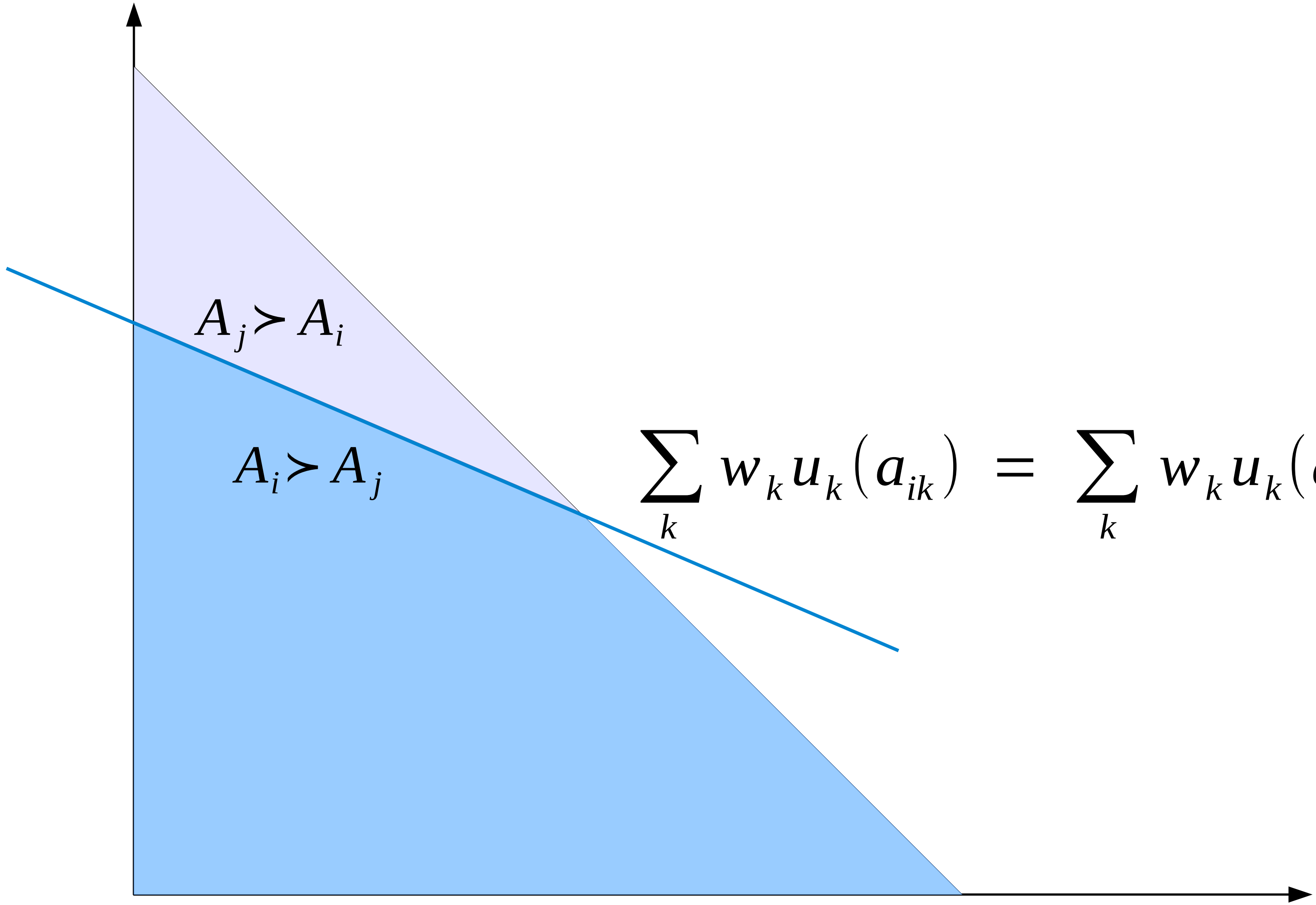
- **Intervals:**
  weight is between....
  $$\underline{w_k} \le w_k \le \overline{w_k}$$
- **Rankings:**
  attribute k is more important than attribute m
  $$w_m \le w_k$$
- **Ratios:**
  attribute k is at least twice as important as m
  $$2w_m \le w_k$$
- Comparison of **alternatives:**
  A_i is better than A_j
  $$\sum_k w_k u_k(a_{ik}) \ge \sum_k w_k u_k(a_{jk})$$

In general: Linear constraints on w_k

Admissible parameters



Example: Constraint from pairwise comparison of alternatives



$$\sum_k w_k u_k(a_{ik}) = \sum_k w_k u_k(a_{jk})$$

Decisions with incomplete information

Approaches

Single parameter:
Identify one "best" parameter vector

- Srinivasan/Shocker 1973
- UTA: Jacquet-Lagreze/Siskos 1982
- Representative value functions: Greco et al. 2011

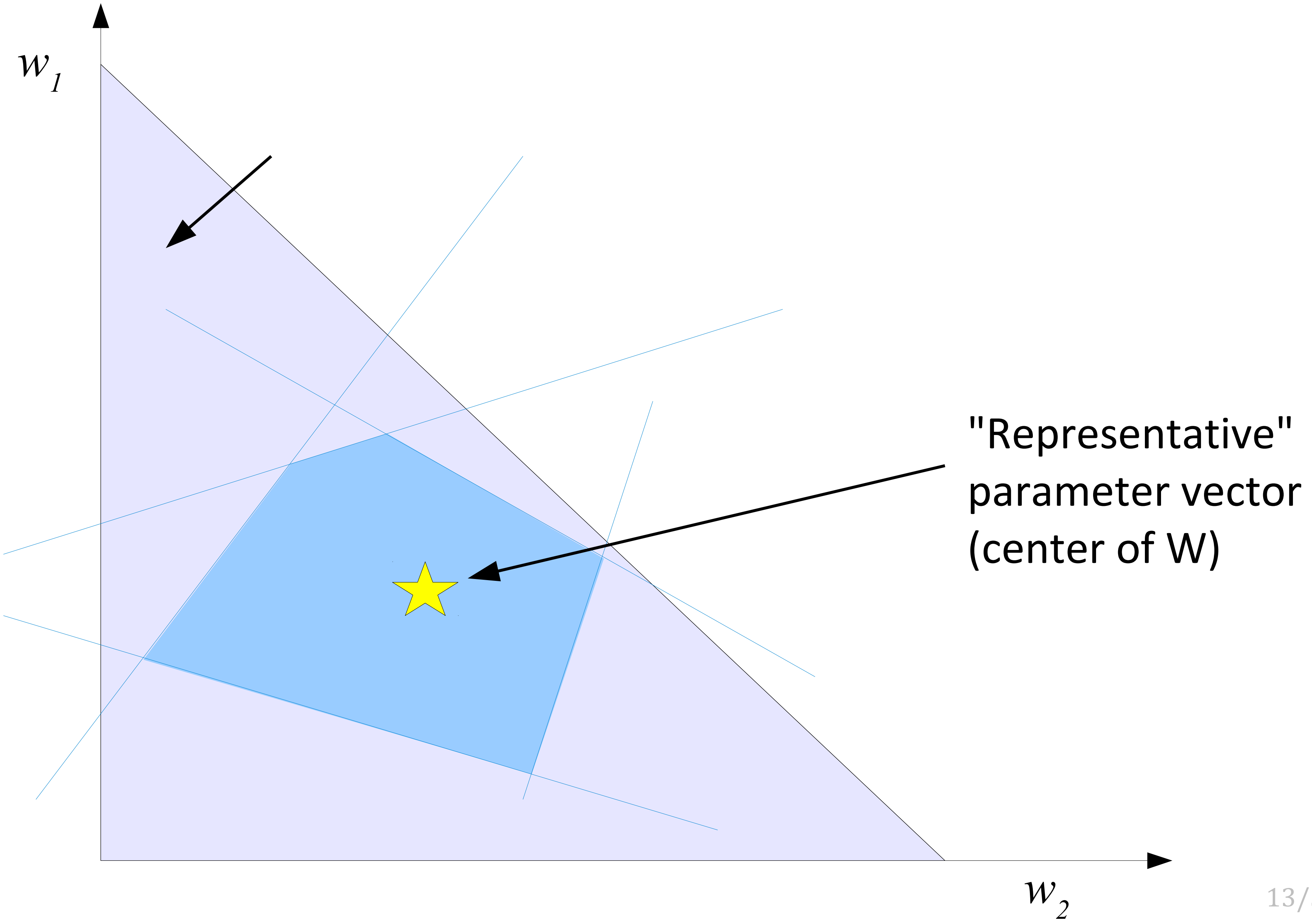
Relation based:
Establish relations that hold for all possible parameters

- Kmietowicz/Pearman 1984
- Kirkwood/Sarin 1985
- Park et al. 1996,1997, 2001
- ROR: Greco et al 2008

Volume-based:
Relative size of regions in parameter space

- Domain criterion: Starr 1962
- Charnetski/Soland 1978
- VIP: Climaco/Dias 2000
- SMAA: Lahdelma et al 1998, 2001

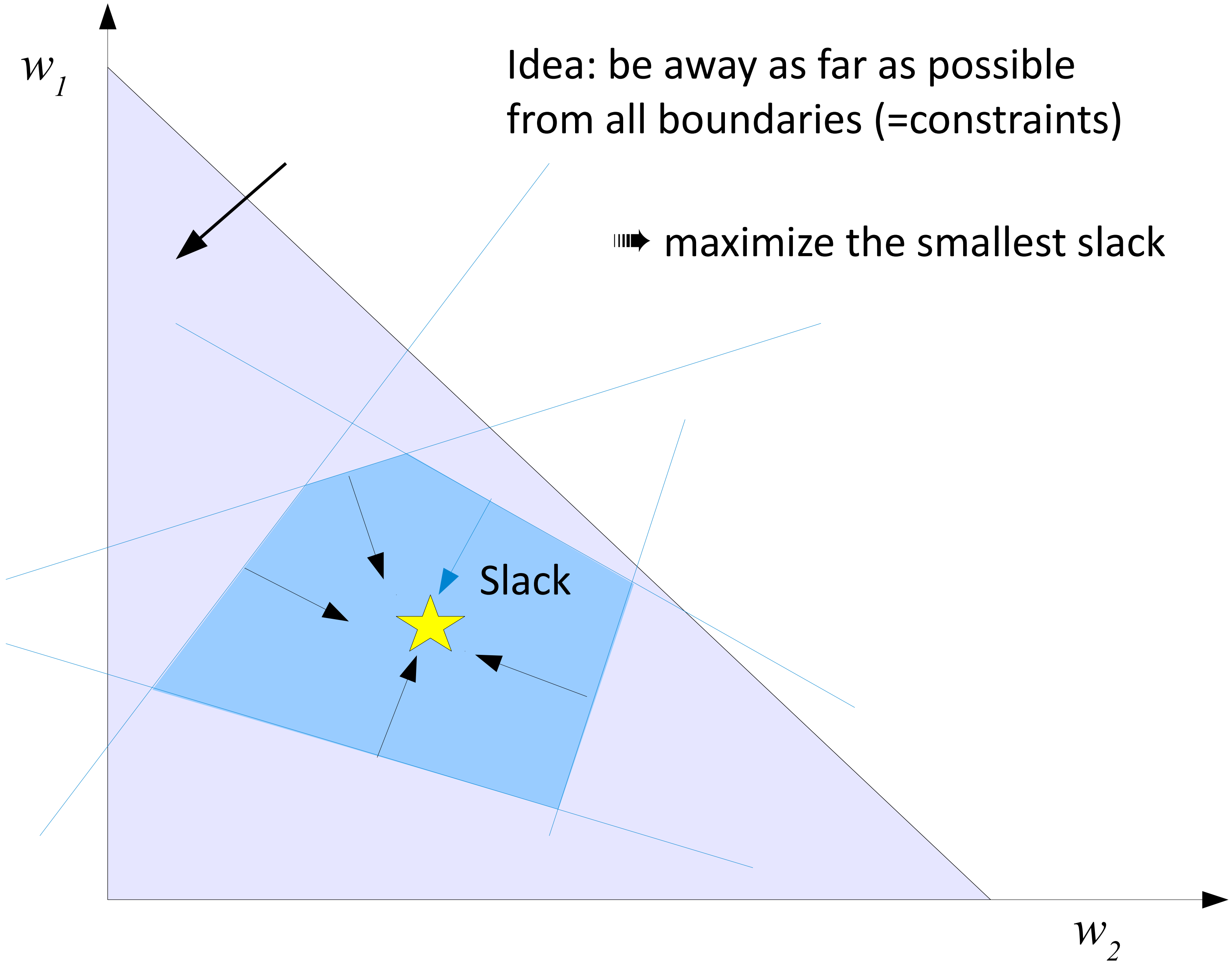
Single parameter approach



Single parameter approach

Idea: be away as far as possible from all boundaries (=constraints)

➠ maximize the smallest slack



Single parameter approach: model

Example: constraints from pairwise comparison of alternatives

$$\max \min z_{ij}$$
$$\text{s.t.}$$
$$\sum_k w_k (a_{ik} - a_{jk}) - z_{ij} = 0 \quad \forall i, j : A_i \succ A_j$$
$$z > 0$$

Single parameter approach: model

Getting rid of the min operator

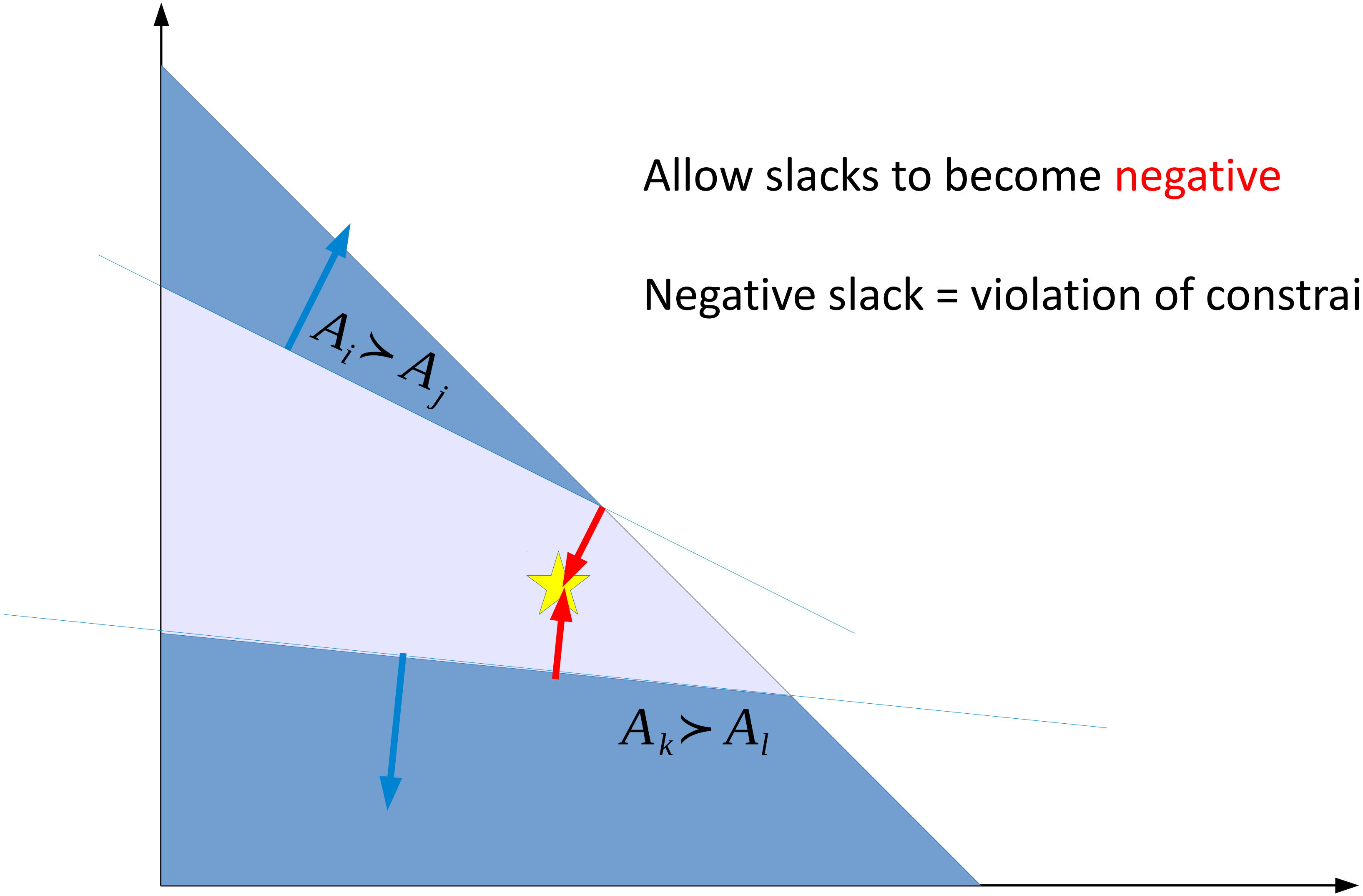
$$\max z$$
$$\text{s.t.}$$
$$z \le z_{ij} \quad \forall i, j : A_i \succ A_j$$
$$\sum_k w_k (a_{ik} - a_{jk}) - z_{ij} = 0 \quad \forall i, j : A_i \succ A_j$$
$$z > 0$$

Single parameter approach: model

And we actually need only one slack

$$
\begin{aligned}
&\max z \\
&\text{s.}t\,. \\
&\sum_k w_k (a_{ik} - a_{jk}) - z \geq 0 \quad \forall i, j : A_i \succ A_j \\
&z > 0
\end{aligned}
$$

What happens if constraints are incompatible?



Allow slacks to become negative

Negative slack = violation of constraints

Single parameter approach: model

Allow for negative slack

$$\max z$$
$$s.t.$$
$$\sum_k w_k(a_{ik} - a_{jk}) - z \geq 0 \quad \forall i,j: A_i \succ A_j$$
$$z \lesseqgtr 0$$

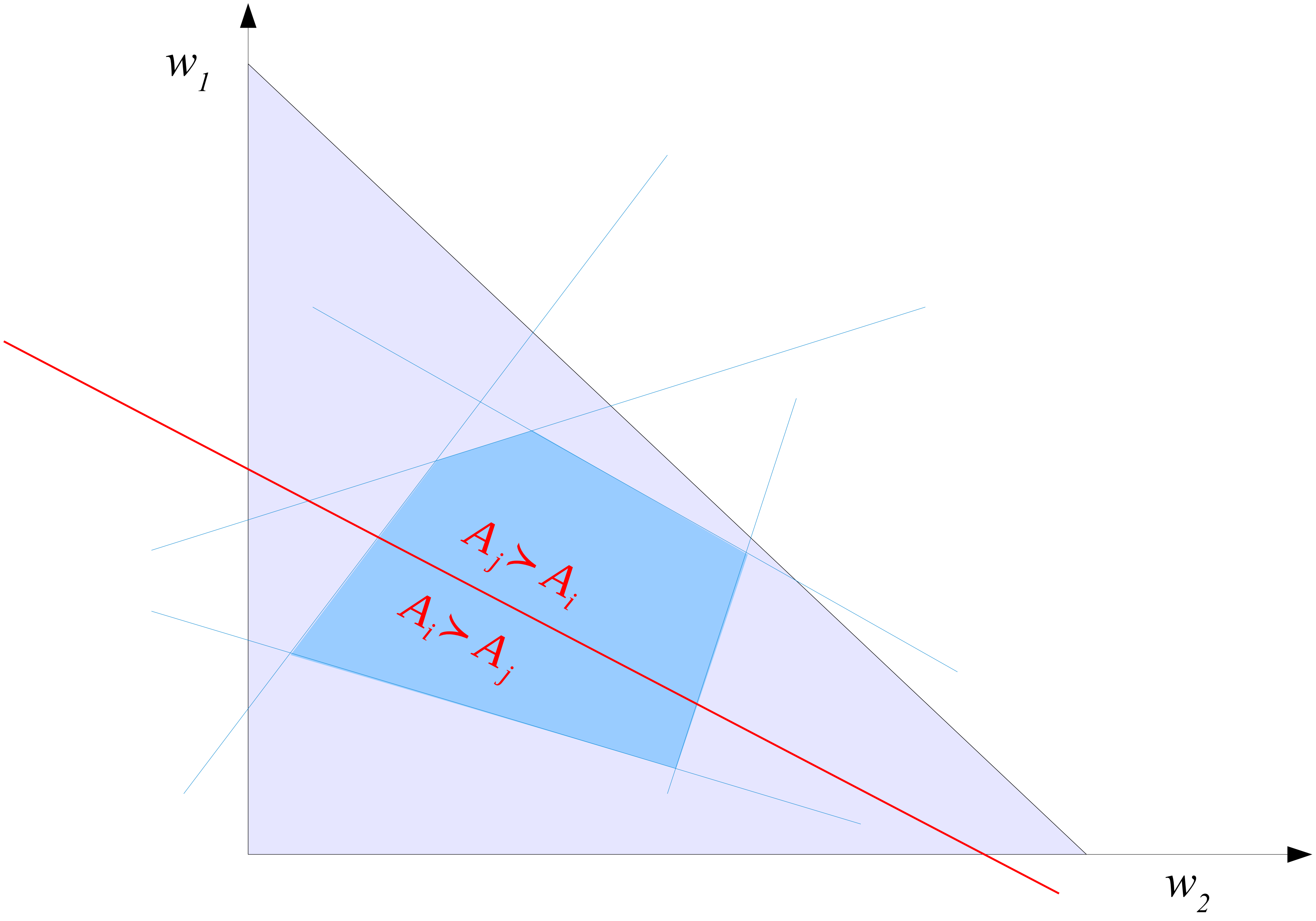
Optimal z is positive: Slack in all constraints, model is compatible with preferences

Optimal z is negative: At least one constraint is violated, model not compatible with preferences

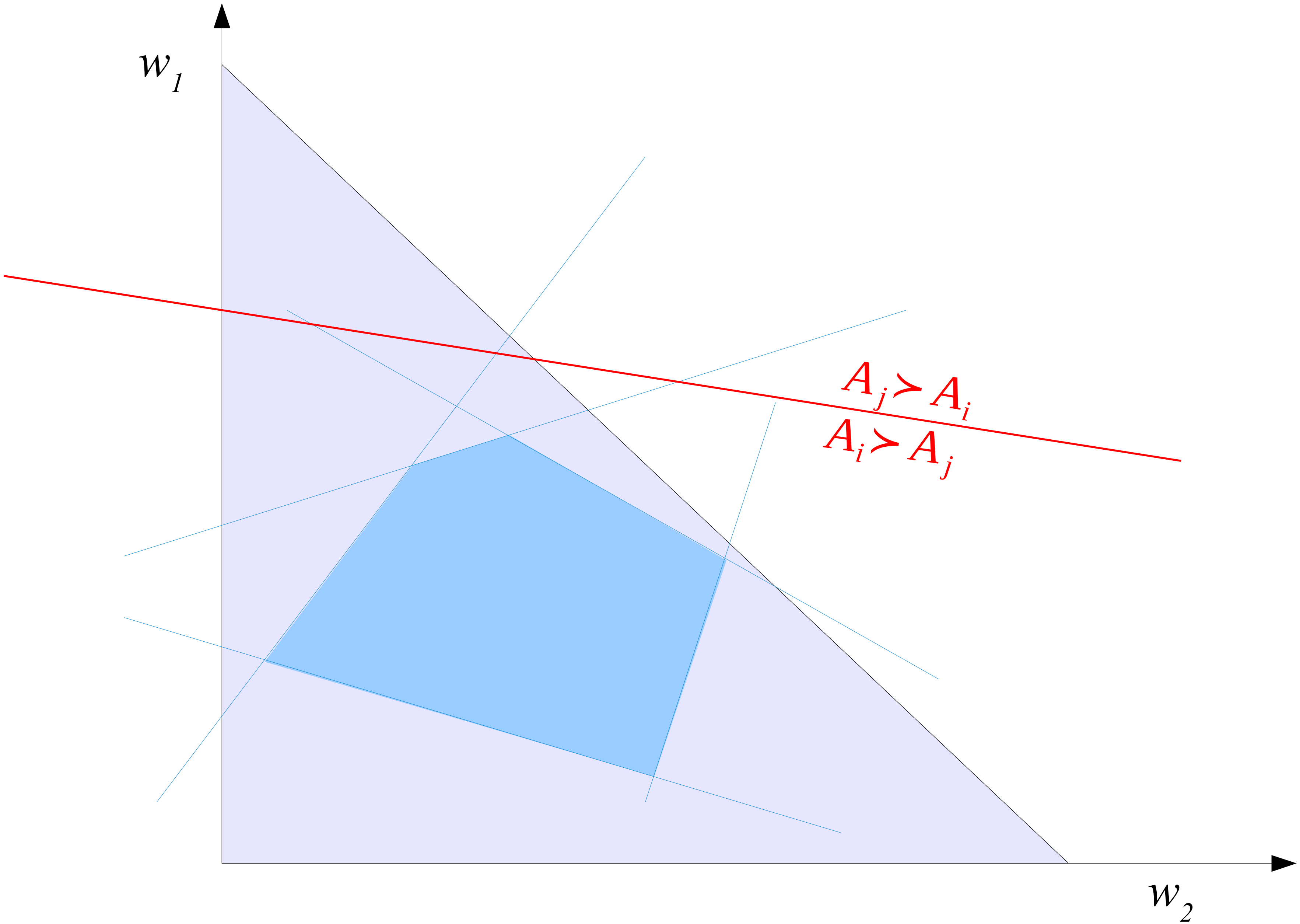
Relation based approach

- Consider preference between two alternatives A_i and A_j
- Can this preference hold, given the information about parameters?
  Possible preference relation
- Will this preference surely hold, given the information about parameters?
  Necessary preference relation

Possible preference



Necessary preference



Testing for necessary preference

$$\max u(A_j|w) - u(A_i|w)$$
$$\text{s.t.}$$
$$w \in W$$

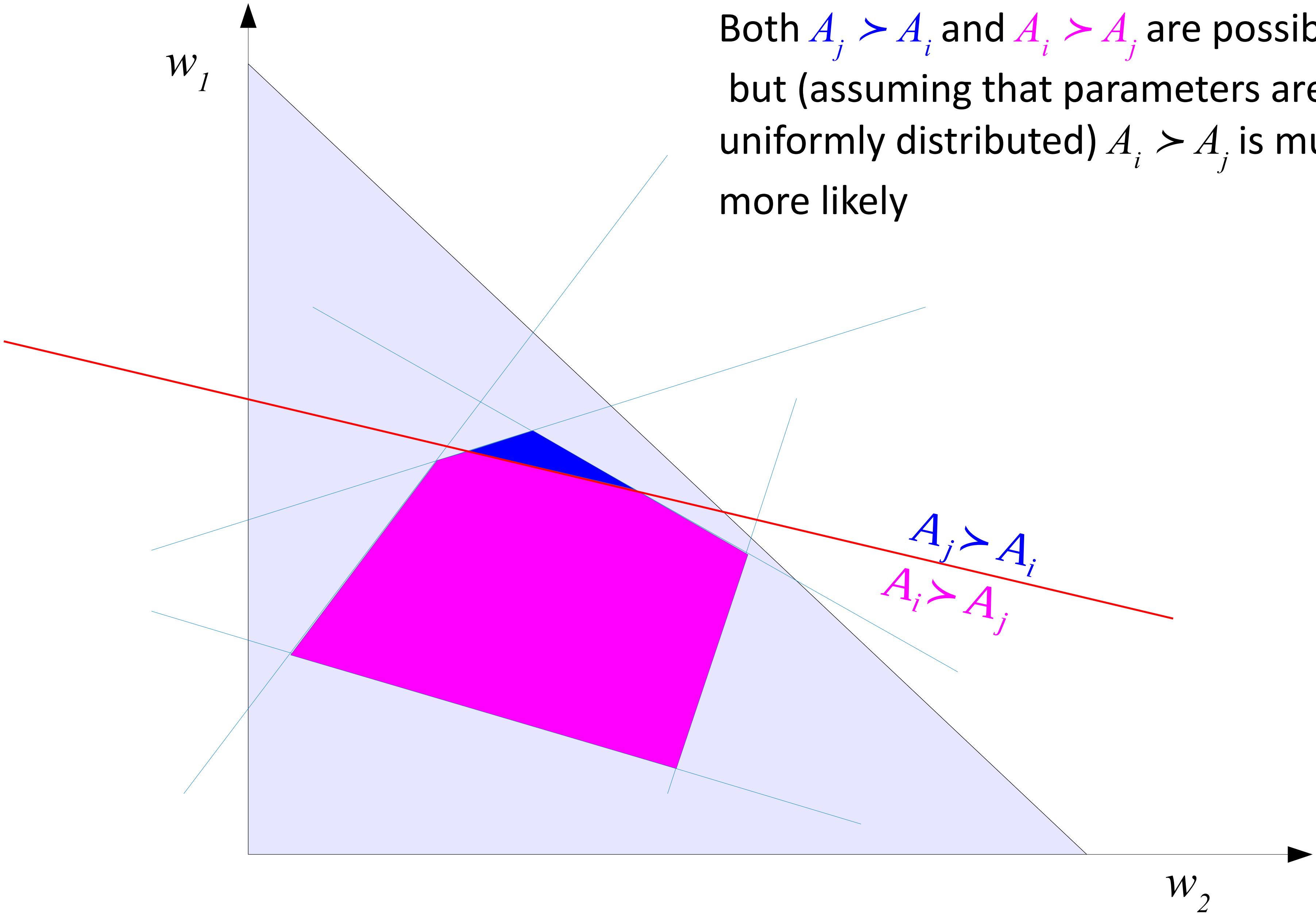
- if optimal objective value is negative
- even in the best case for A_j, it is worse than A_i
- A_i is necessarily preferred to A_j

Relation based approaches

- Necessary preference is subset of possible preference
- Necessary preference is typically incomplete
- Possible preference is often inconclusive (holds in both directions)
- Relation "in between" could be useful

Another case of possible preference

Both $A_j \succ A_i$ and $A_i \succ A_j$ are possible, but (assuming that parameters are uniformly distributed) $A_i \succ A_j$ is much more likely

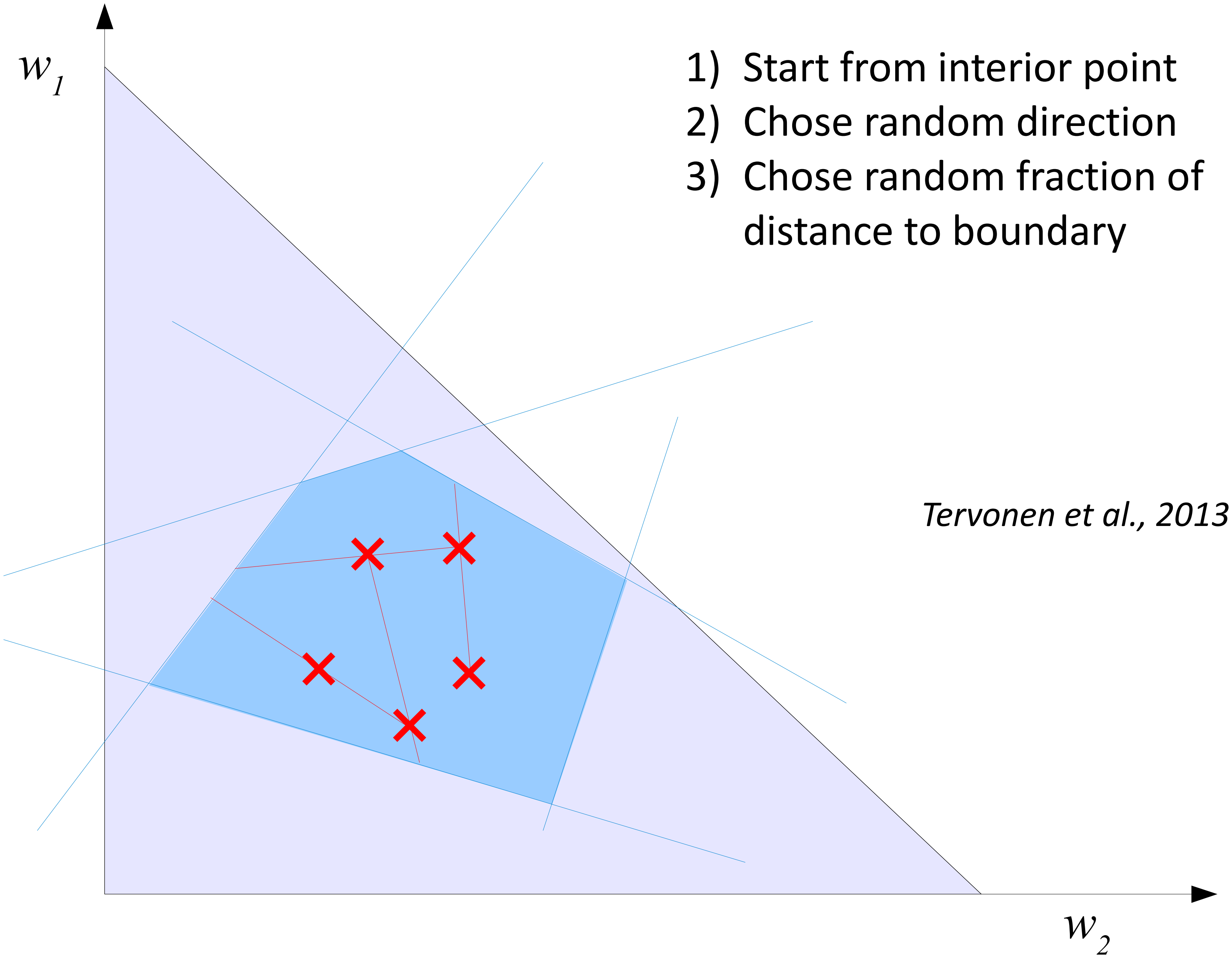


Volume-based approach: SMAA – **S**tochastic **M**ultiattribute **A**cceptability **A**nalysis

- Assume that parameters are uniformly distributed
- Volume of subset of parameter space as probability
- Use to estimate probabilities of certain facts to hold
  - an alternative is best
  - an alternative has a certain rank in the ranking of all alternatives
  - an alternative is ranked better than another alternative
- Usually done by simulation

Lahdelma, 1998

Sampling in constrained sets: "Hit and Run" method



1) Start from interior point
2) Chose random direction
3) Chose random fraction of distance to boundary

Tervonen et al., 2013

Results from volume-based methods (SMAA)

- **Rank acceptability** index:
  Probability r_{ik} that alternative A_i obtains rank k
- **Pairwise winning** index:
  Probability p_{ij} that alternative A_i is preferred to A_j
- How to transform into a ranking of alternatives?

Model for rank acceptability index

Assignment problem

$$\sum_{k=1}^{N_{alt}} x_{ik} = 1 \quad \forall i$$

each alternative is assigned to one rank

$$\sum_{i=1}^{N_{alt}} x_{ik} = 1 \quad \forall k$$

to each rank, one alternative is assigned (omit to allow indifference)

$$x_{ik} \in \{0,1\}$$

x_{ik}: Alternative A_i is assigned to rank k

Objective functions

Average probability of assignments

$$\max \sum_{i,k:x_{ik}=1} r_{ik} = \sum_{i=1}^{N_{alt}} \sum_{k=1}^{N_{alt}} r_{ik} x_{ik}$$

Joint probability

$$\max \prod_{i,k:x_{ik}=1} r_{ik} = \sum_{i=1}^{N_{alt}} \sum_{k=1}^{N_{alt}} \log(r_{ik}) x_{ik}$$

Minimum probability of assignment

$$\max z$$
$$z \le r_{ik} + (1 - x_{ik}) \quad \forall i, k$$

Models for pairwise winning indices

Construct complete order relation:

Complete and asymmetric

$$y_{ij} + y_{ji} = 1 \quad \forall i \neq j$$

Irreflexive

$$y_{ii} = 0 \quad \forall i$$

With indifference

$$y_{ij} + y_{ji} \geq 1 \quad \forall i \neq j$$

$$z_{ij} = y_{ij} + y_{ji} - 1$$

Transitive

$$y_{ij} \geq y_{ik} + y_{kj} - 1.5 \quad \forall k \neq i, j$$

y_{ij}: Alternative A_i is preferred to (at least as good as) A_j

z_{ij}: Indifference between A_i and A_j

Linking models

Rank from assignment model

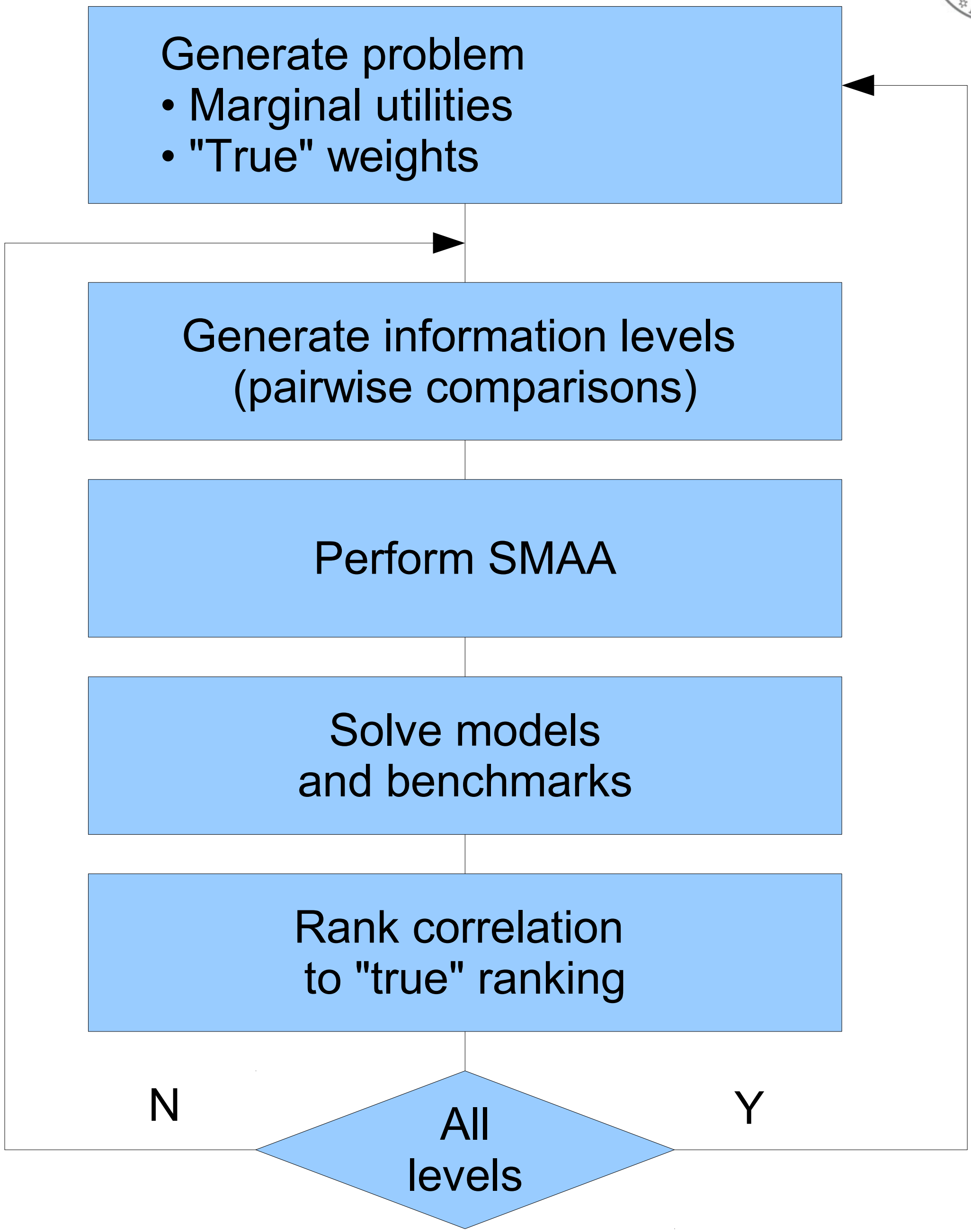
$$R_i = \sum_k k\, x_{ik}$$

Rank from relation model

$$R_i = 1 + \sum_j y_{ji}$$

=

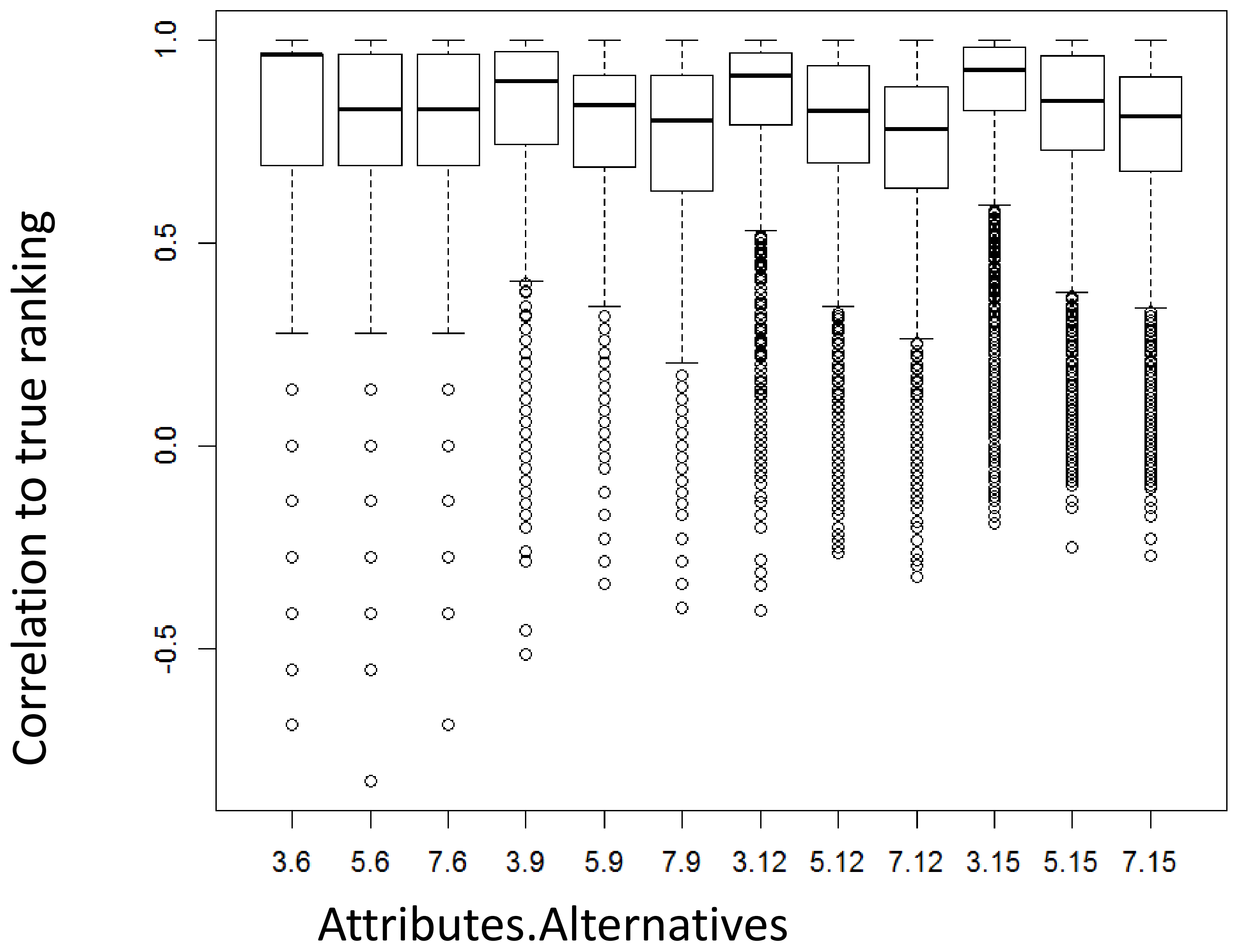
Computational study



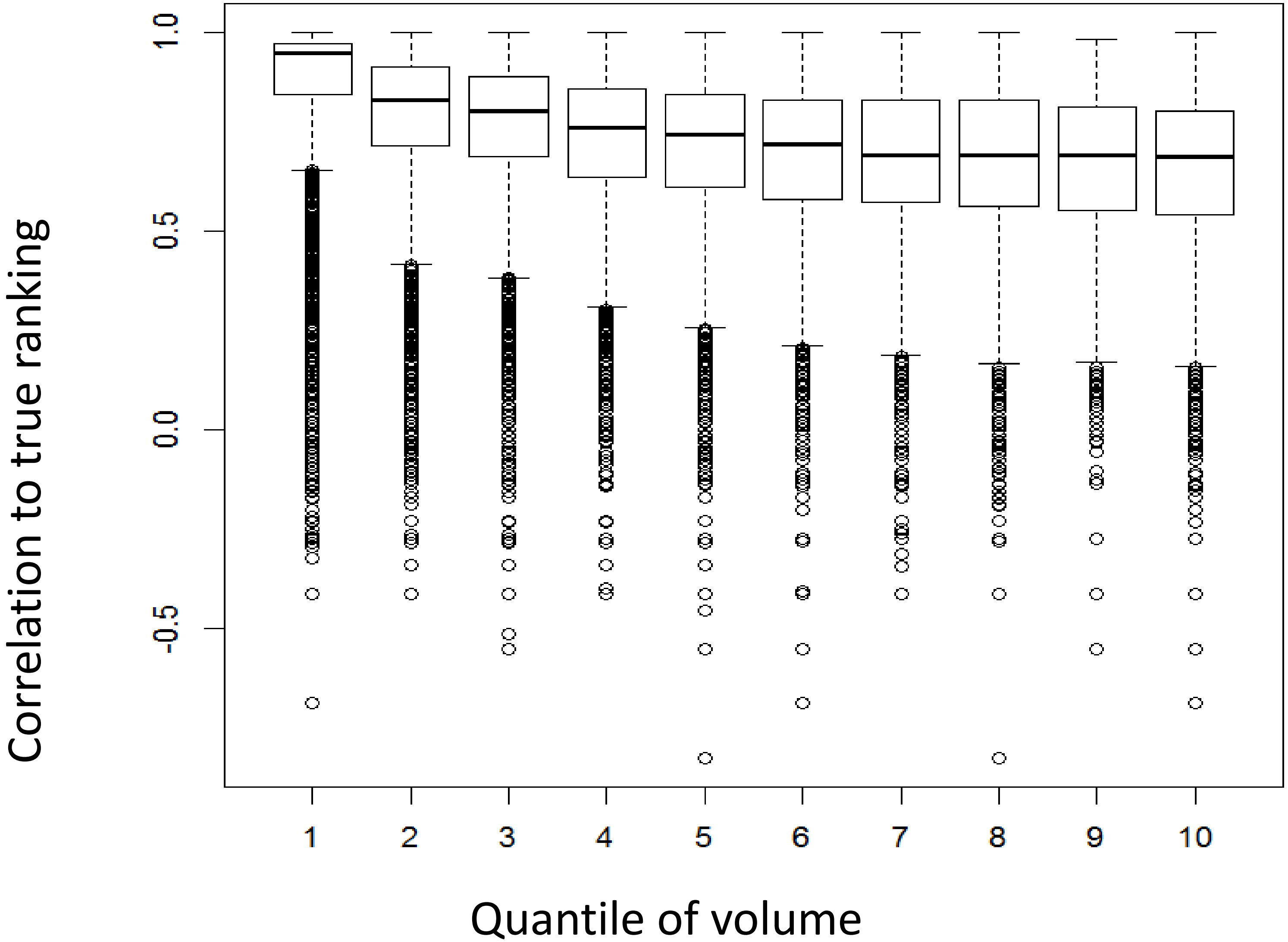
Computational study

- Problem dimensions:
  - 3, 5, 7 attributes,
  - 6, 9, 12, 15 alternatives
- 2 methods for generating comparisons (by alternative# and by "true" ranking)
- available information: $\mathrm{Vol}(W)/\mathrm{Vol}(\mathit{Unit\ simplex})$
- 500 randomly generated problems for each problem dimension and information method

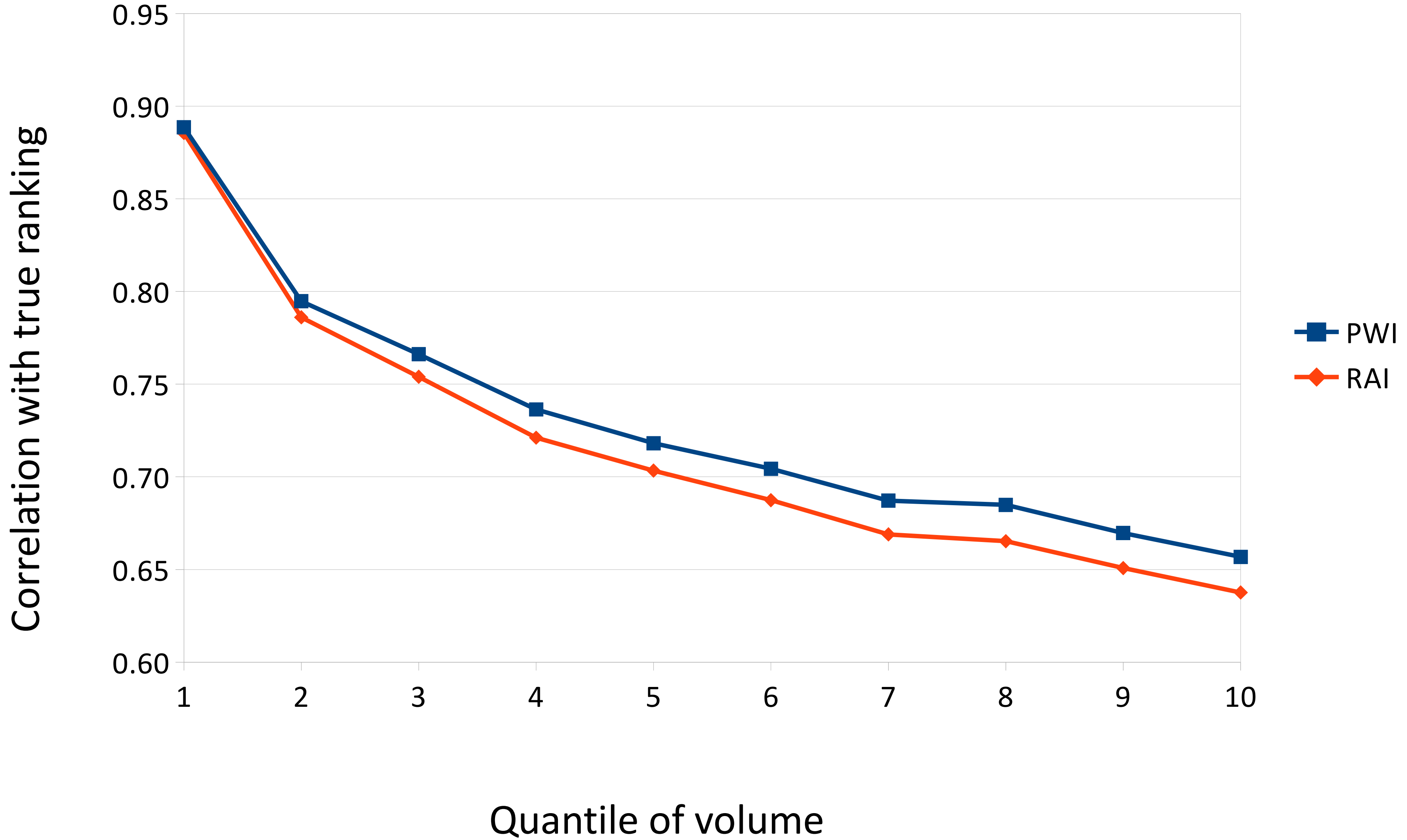
Effects of problem dimensions



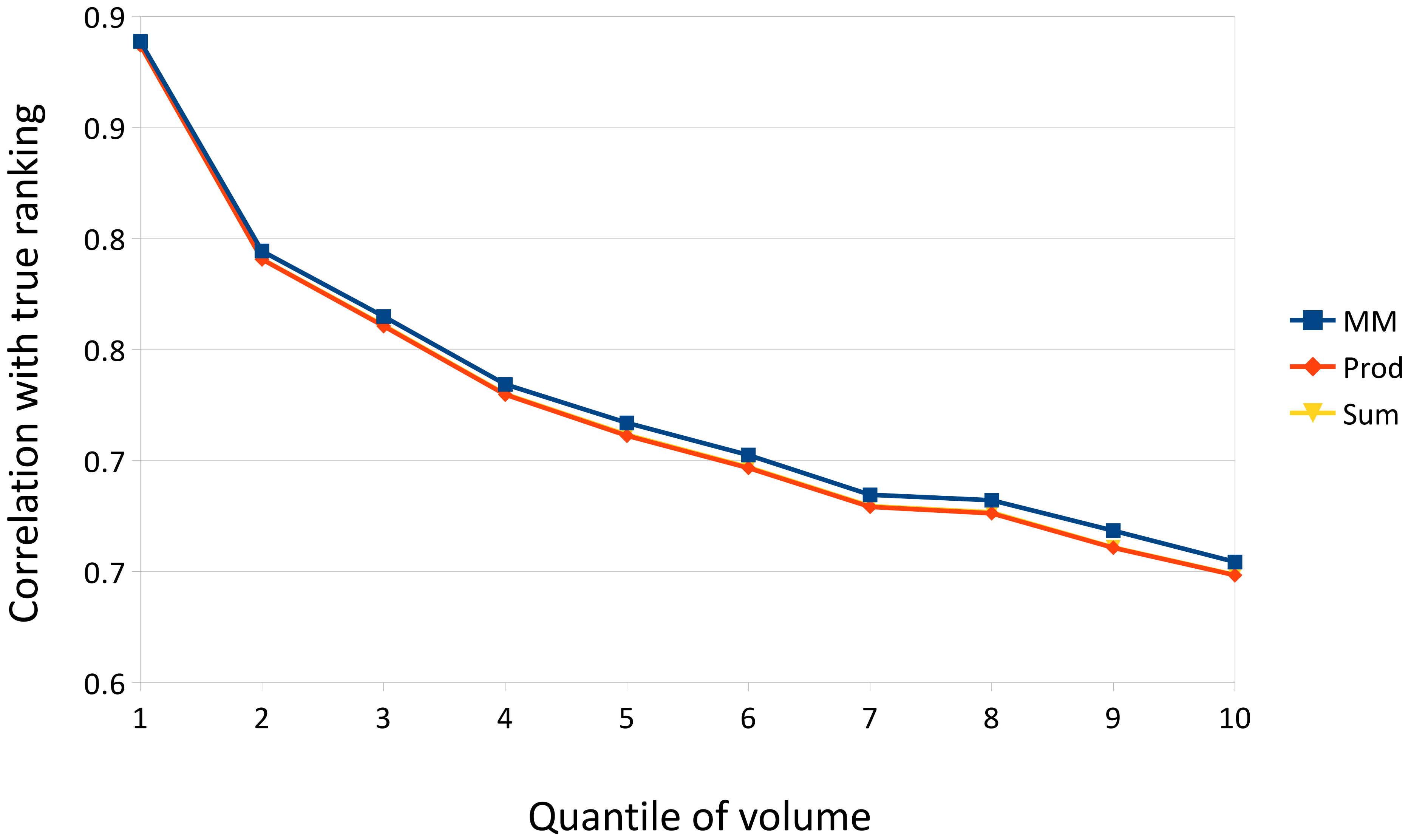
Effect of information



Differences between indices



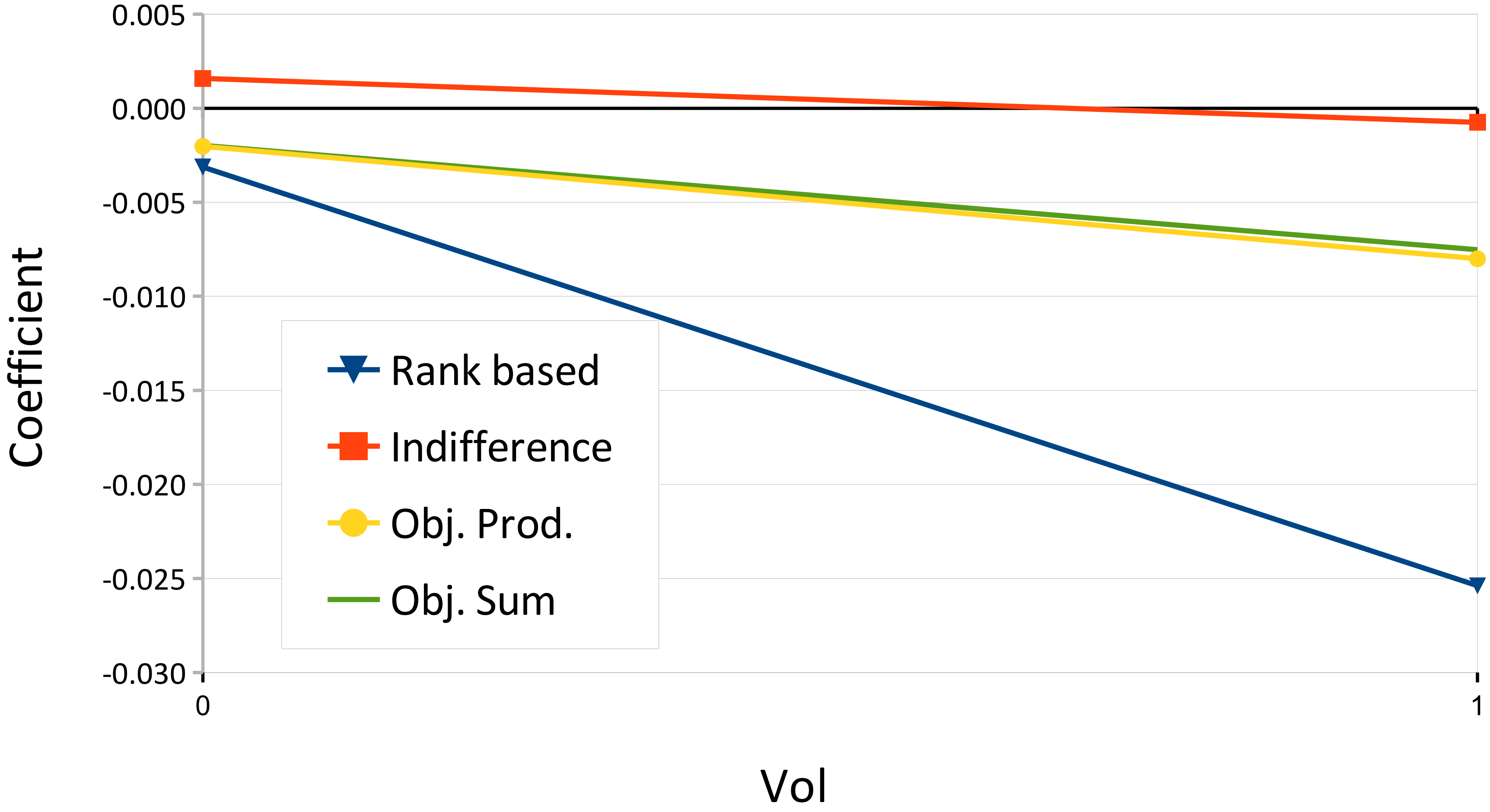
Differences between objective functions



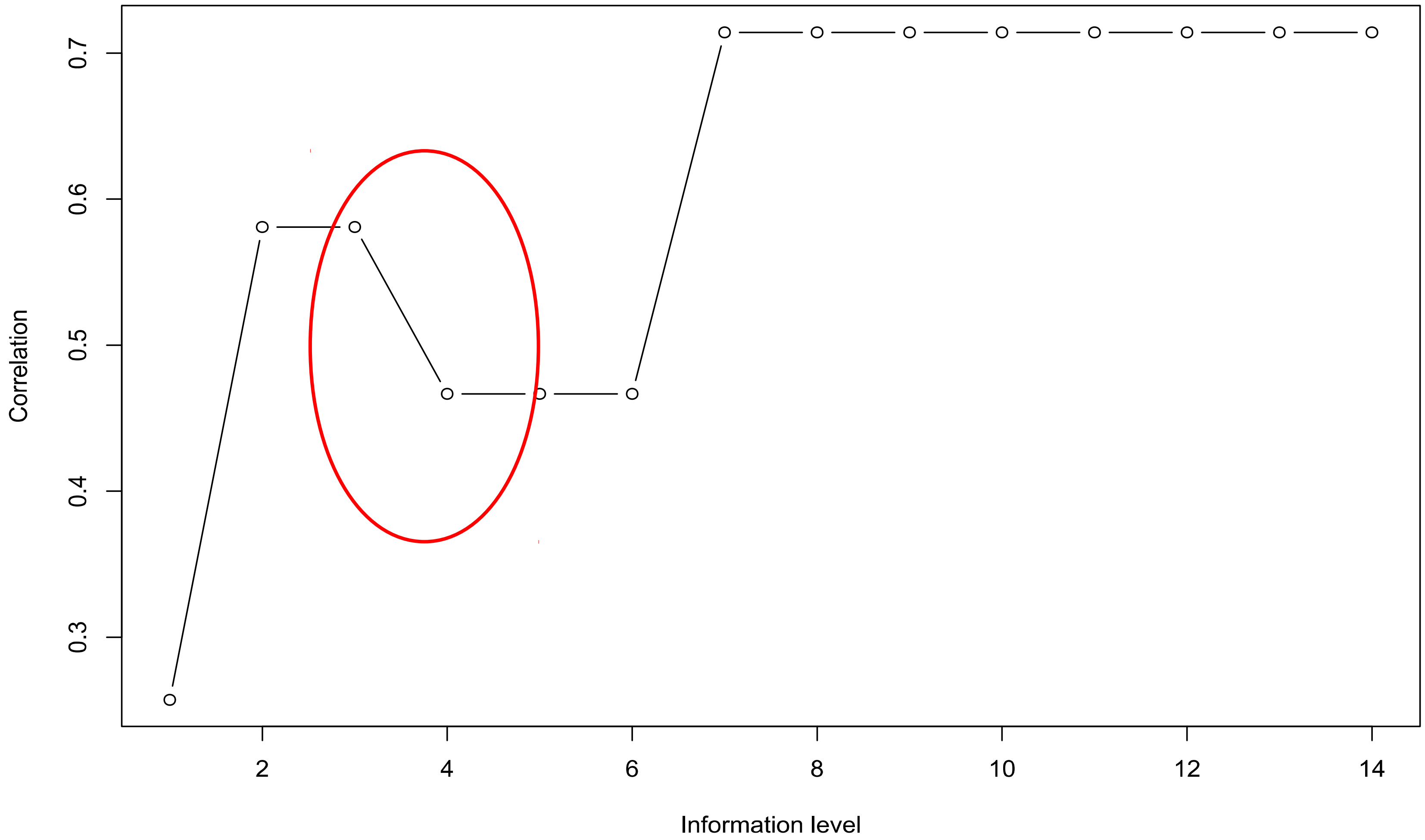
Regression model

	M1	M2	M3	M4
(Intercept)	*** 0.8372	*** 0.9752	*** 0.9799	*** 0.9770
NAlt=9	*** 0.0034	*** -0.0327	*** -0.0327	*** -0.0327
Nalt=12	*** 0.0112	*** -0.0430	*** -0.0430	*** -0.0430
Nalt=15	*** 0.0355	*** -0.0325	*** -0.0325	*** -0.0325
Ncrit=5	*** -0.0583	*** -0.0756	*** -0.0756	*** -0.0756
Ncrit=7	*** -0.0900	*** -0.1136	*** -0.1136	*** -0.1136
Vol		*** -0.3216	*** -0.3216	*** -0.3096
Rank based			*** -0.0084	*** -0.0031
Indifference			** 0.0010	*** 0.0016
Obj. Prod.			*** -0.0034	*** -0.0020
Obj. Sum			*** -0.0033	*** -0.0020
Interaction Vol with ...				
Rank based				*** -0.0223
Indifference				° -0.0023
Obj. Prod.				*** -0.0060
Obj. Sum				*** -0.0055
R2	0.0490	0.2772	0.2778	0.2781

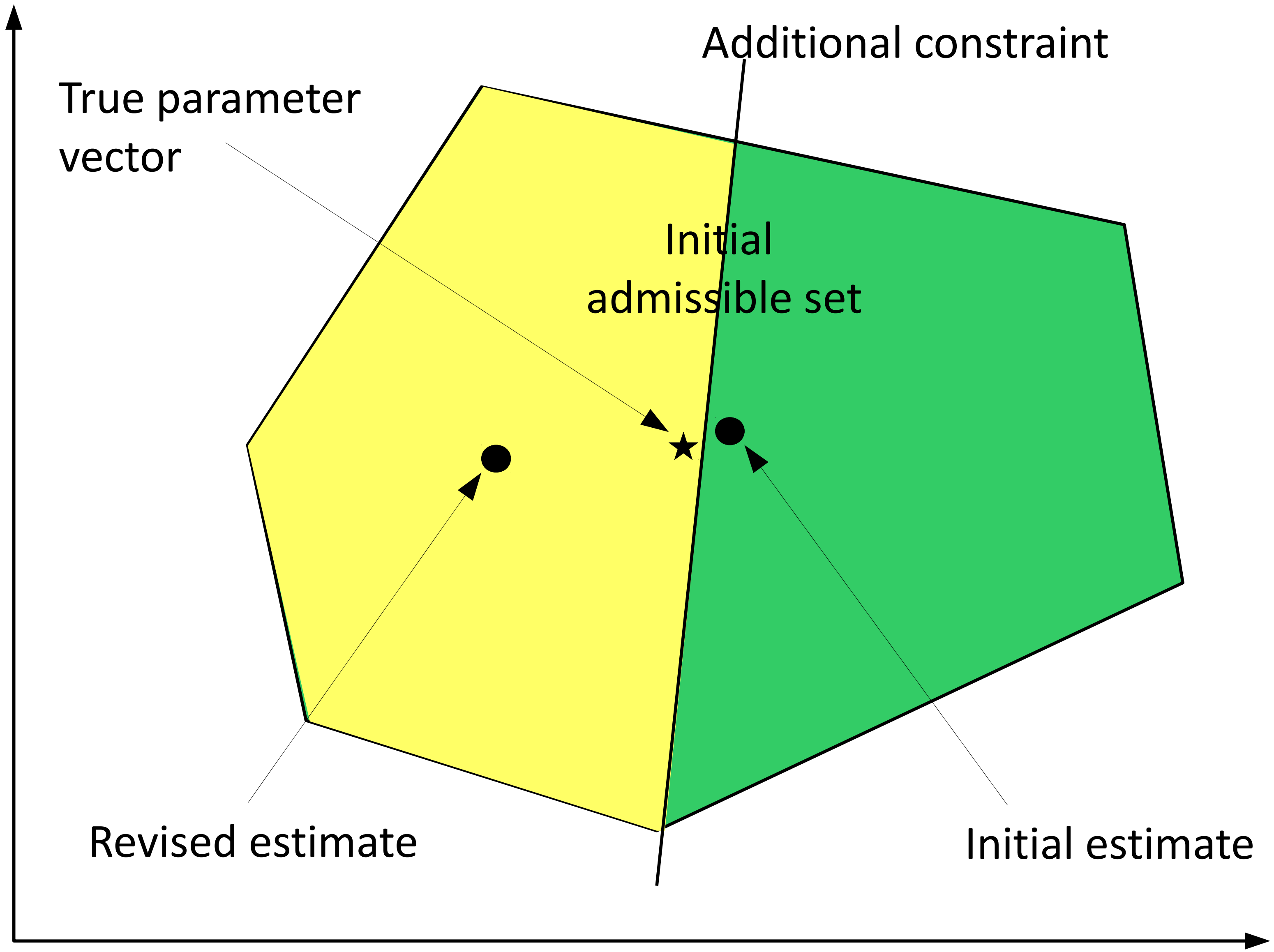
Interpretation of interaction terms



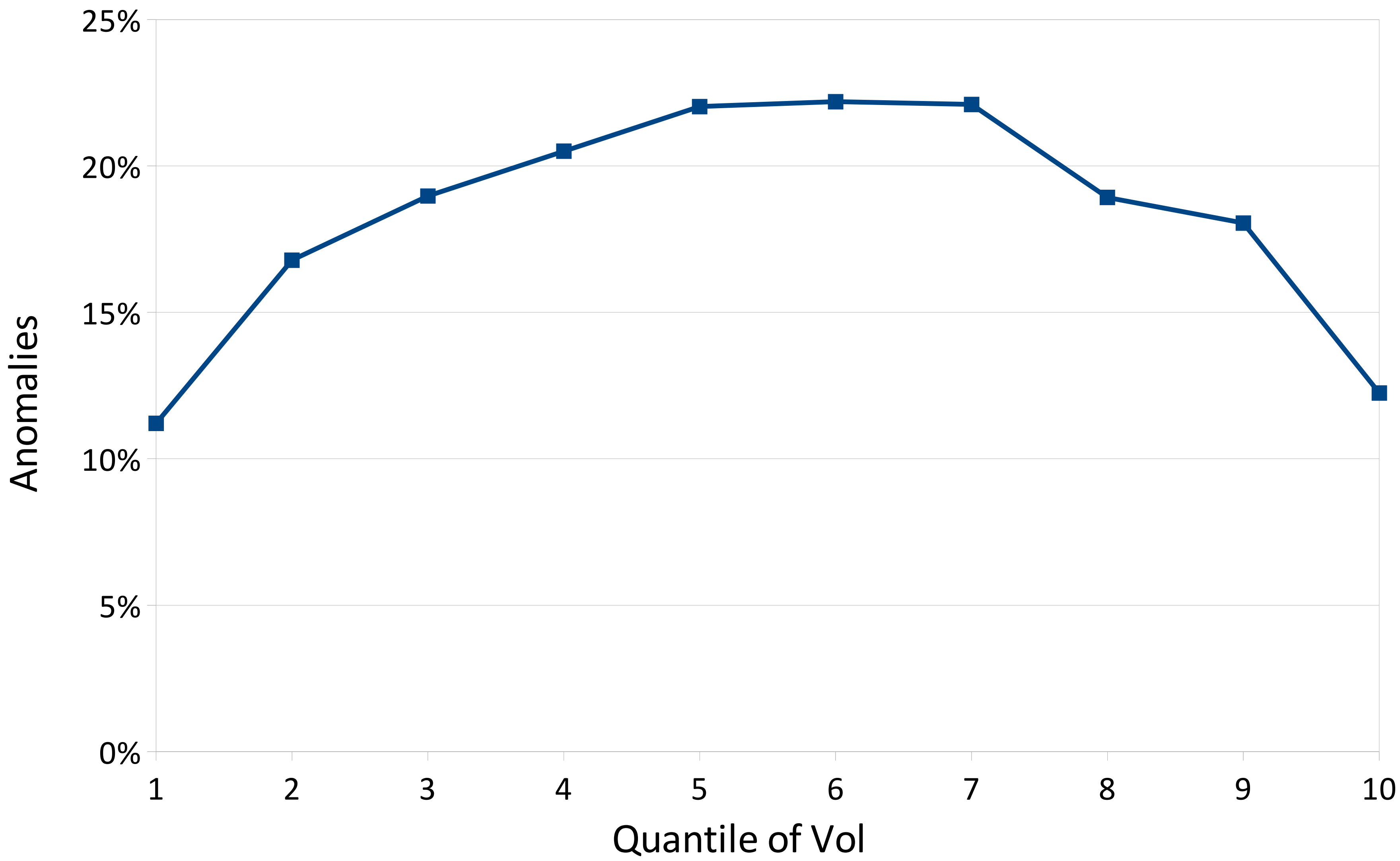
Information in one experiment



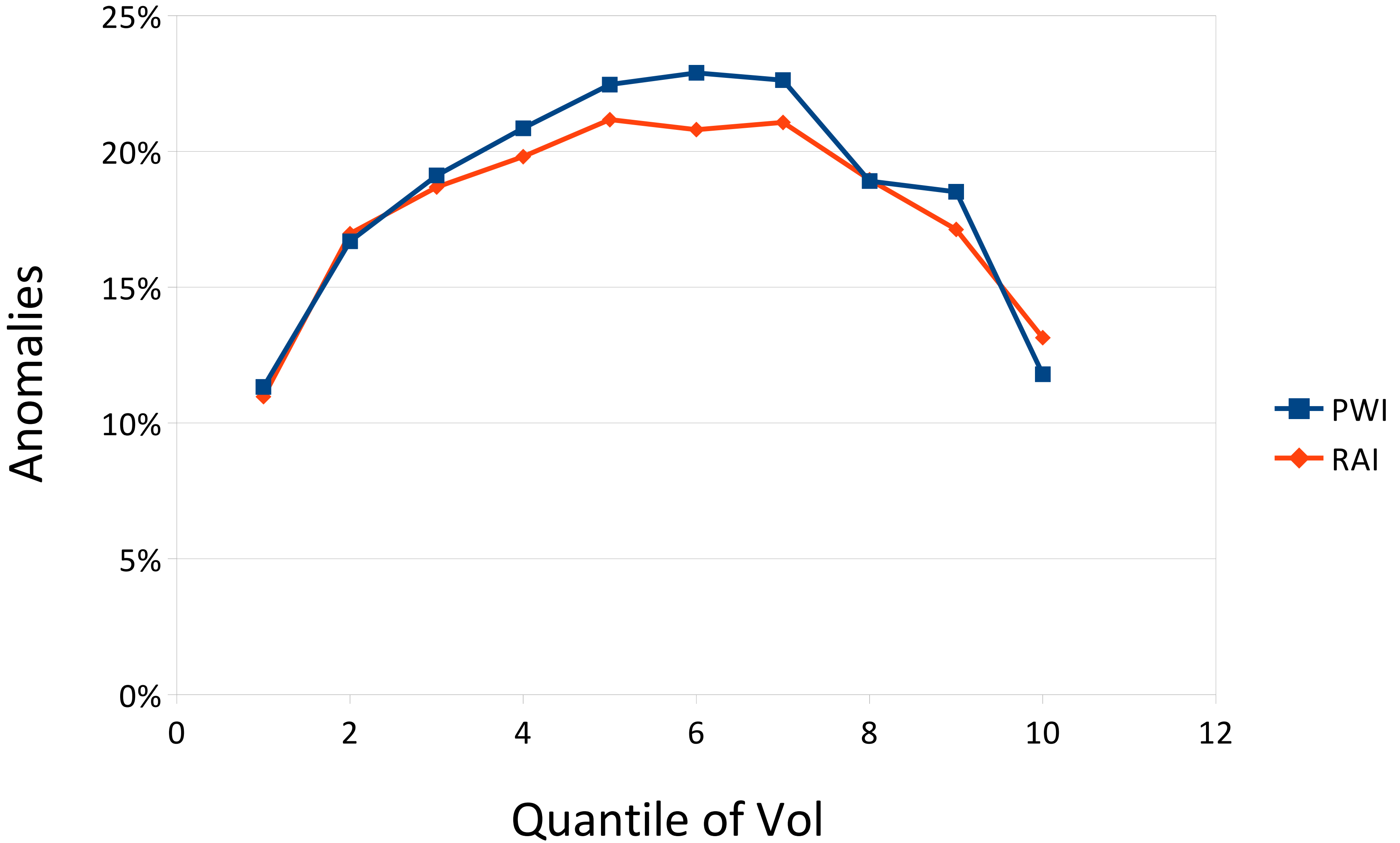
Information anomalies



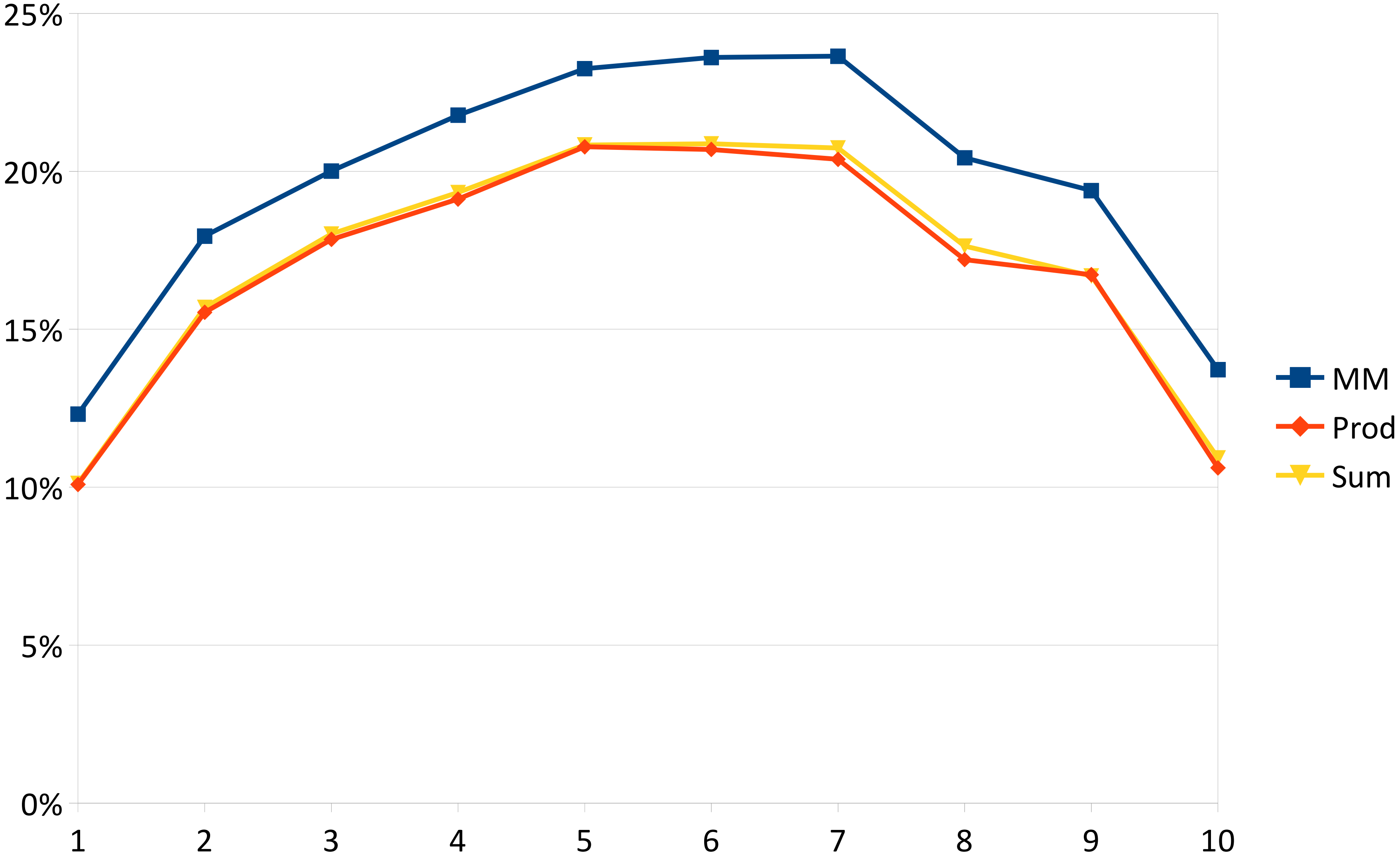
Occurrence of information anomalies



Differences between indices



Differences between objective functions

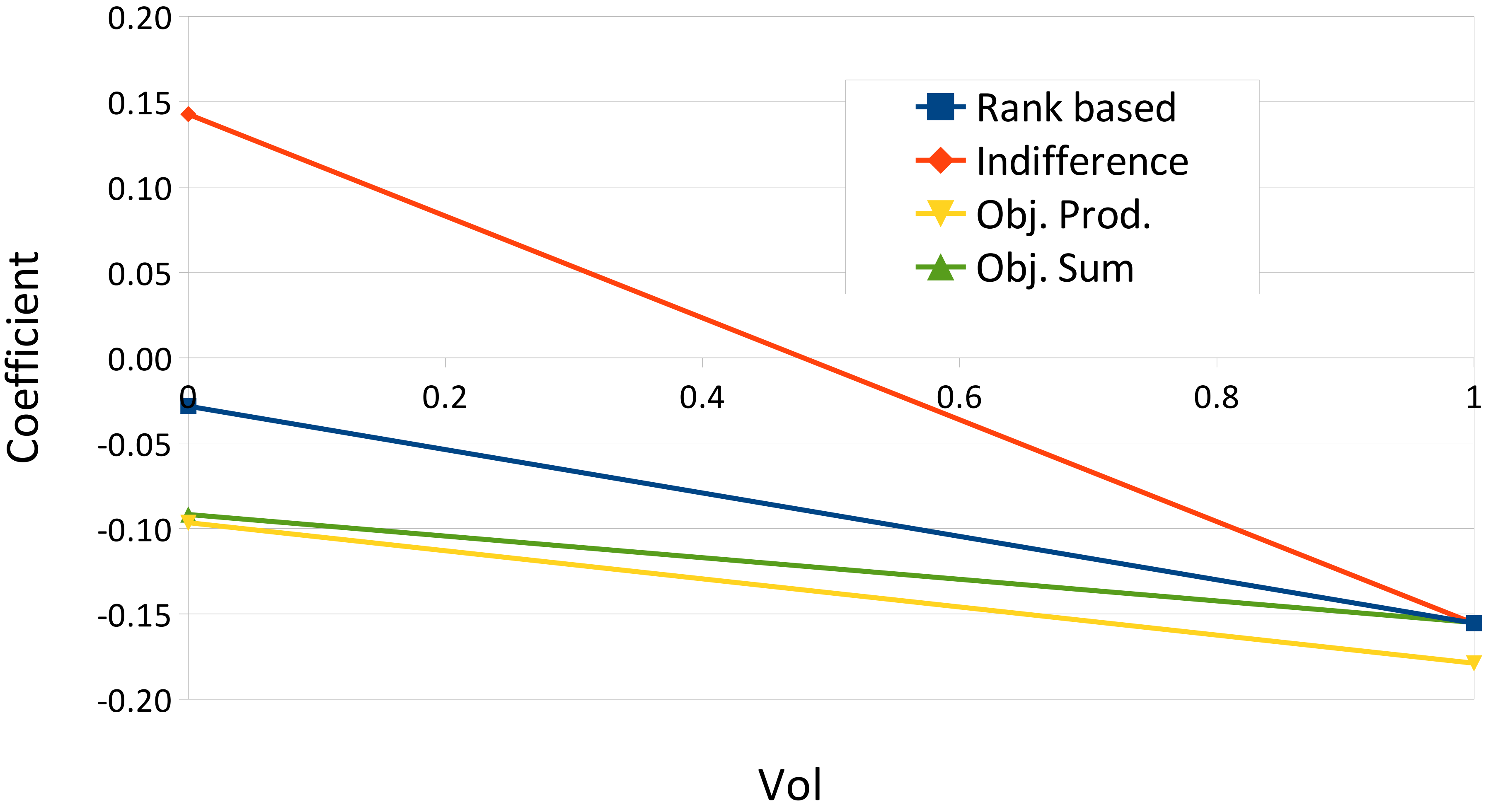


Regression results

	m1	m2	m3	m4
(Intercept)	*** -2.9947	*** -3.8814	*** -3.8171	*** -3.8609
Nalt=9	*** 0.5923	*** 0.7885	*** 0.7890	*** 0.7890
Nalt=12	*** 0.8712	*** 1.1917	*** 1.1925	*** 1.1926
Nalt=15	*** 1.0313	*** 1.4559	*** 1.4570	*** 1.4571
Nkrit=5	*** 0.4479	*** 0.5827	*** 0.5831	*** 0.5831
Nkrit=7	*** 0.6632	*** 0.8544	*** 0.8551	*** 0.8552
Vol		*** 3.5900	*** 3.5930	*** 3.7857
Vol 2		*** -5.1733	*** -5.1775	*** -5.1834
Rank based			*** -0.0575	** -0.0283
Indifference			*** 0.0753	*** 0.1427
Obj. Prod.			*** -0.1153	*** -0.0966
Obj. Sum			*** -0.1061	*** -0.0918
Interaction Vol..				
Rank based				*** -0.1271
Indifference				*** -0.2983
Obj. Prod.				** -0.0824
Obj. Sum				* -0.0633
AIC	776534.83	746548.90	745912.25	745819.61

Logistic regression on occurrence of anomaly

Interpretation of interactions



Summary of methods

Approaches

Single parameter:
Identify one "best" parameter vector

- Low effort
- Loss of information about uncertainty
- Well-defined ranking

Relation based:
Establish relations that hold for all possible parameters

- Medium effort
- No clear ranking
- Incomplete relation
- Most robust decision

Volume-based:
Relative size of regions in parameter space

- High effort (simulation)
- Rich information
- No clear ranking

↓

Complete ranking via models presented

Conclusions

- Uncertainty of parameters is important for realistic decision models
- Different approaches available:
  - Single parameter vector
  - Relation-based
  - Volume-based
- Represent a scale between richness of information and effort
- New approaches to generate rankings from probabilistic information

References

- Single parameter
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Thank you for your attention!