Optimal Inventory-allocation Integrated Model for Perishable Items with Stochastic Demand in a Single-vendor Multi-retailer Supply Chain

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Abstract

This study investigates an increasingly widespread supply chain system that involves a vendor cyclically supplying specific perishable item to multiple retailers. In general, the vendor has to earlier and accurately determine inventory quantity to meet aggregate demand, and exactly allocate the inventory among a number of retailers to reduce the adjustment costs. Meanwhile, a vital challenge faced for vendors is to develop an efficient inventory-allocation decision model for given perishable item with stochastic and correlated retailer demands during upcoming selling period. To this end, an effective and practical analytical approach, which extends the newsvendor model to incorporate the considered inventory-allocation decision, is proposed here to simultaneously solve the optimal inventory quantity and allocation policy for maximizing expected vendor profits of perishable items. Especially, the lognormal distribution and Ito process is used here to model the behaviors of individual demand shift. Also, an effective integrated approach is presented to work out the aggregate demand during next selling period. Finally, numerical experiments are conducted to demonstrate and validate the proposed model and extract the valuable managerial findings.

Keywords: Inventory-allocation model, single-vendor multi-retailer supply chain, newsvendor model, perishable item, stochastic demand.

1. Introduction

The inventory policies in traditional supply chain systems are mostly dominated and determined by downstream retailers that are motivated for maximizing their profits. As a result, most studies on supply chain management and inventory control focused on investigating the economic ordering model from buyer’s perspective. However, recently some business patterns activating an emerging tendency in supply chains is for the vendor to manage inventory such that the supplier or manufacturer acts as a leader and trigger of the supply chain and gets a predominant position from the retailers to help optimize wholesales. Among others, two typical systems are as follows: (1) Vendor concurrently serves as retailer and owns the retail stores. (2) In a consignment stock supply chain
retailers cooperate with vendor and simply provide a selling site and warehouse in order for hedging the risk of sales, earn sales commission, and possibly receive compensation for inventory costs. In summary, this study looks at inventory decision of vendor in a supply chain and would be applicable to most vendor-managed inventory practices.

Besides vendor-managed inventory, the study considers and analyzes a common two-echelon supply chain system, which consists of a single vendor and multiple retailers and involves periodic supply and sale of given perishable item. Furthermore, vendor produces and delivers period by period a quantity corresponding to expected aggregate demand for given perishable item during the upcoming selling period and suitably allocate the inventory among multiple retailers. In practice, perishable items are common and necessary in everyday life, such as electronic components, fashion items, foodstuffs, beverages, pharmaceuticals, chemicals, printing goods, and so on. A crucial challenge for vendors in making an accurate and profitable inventory-allocation decision is subject to numerous volatile demands sourced from individual retailer and correlated demands between any two retailers during a selling period. If the accurate inventory-allocation decision is made by vendor, implying that the loss of unsold inventory, inventory expenses, shortage cost, and adjustment cost can be jointly reduced, and thus being expected to sizably improve vendor profitability and customer service level. Correspondingly, this study aims for developing a practical single-period inventory-allocation decision model for vendor of given perishable item in a single-vendor multi-retailer supply chain.

To this end, the production/delivery quantity and allocation of the given perishable item meets the real demand of each retailer in the supply chain as closely as possible. Furthermore, precise estimates of aggregate demand and individual demands derived from retailers become critical when unsold inventory remains little salvage value and adjustment costs are costly among retailers. Therein, adjustment cost represents a cost incurred by the shunting operation that delivers the item of surplus retailer to shortage retailer.

Inventory issues involving perishable items, which are characterized by over a finite selling horizon, is considered frequently as newsvendor problems and completely perishable and deteriorating inventory issues have received significant attention and been widely investigated in literature. Several representative researches on tackling survey of literature on perishable/deteriorating inventory rendered the overview of development trend. These contributions mainly were made by Nahmias [35], Raafat [36], Goyal and Giri [17], Khanlarzade et al. [29], and Janssen et al. [27]. Based on the works of their researches, the investigating issues for perishable/deteriorating inventory can been approximately generalized into a number of central study fields including, for example, review system, replenishment cycle, pricing policy, newsvendor and EOQ model, inventory management with a deteriorating rate, deterministic/probabilistic demand, perishable/deteriorating items with a fixed or random lifetime, returns policies, single product and multiple products, and single-period and multiple period ordering model. Among these fields, this study can be positioned as a single-period newsvendor model for a single perishable item with probabilistic demand and fixed lifetime.
Typical researches on single-vendor multi-buyer supply chain models include as follows. Ingene and Parry [26] and Chen et al. [12] put forward coordination mechanisms for a distribution system involving one supplier and multiple retailers to optimize channel-wide profits. Kim et al. [30] proposed an analytical model to integrate and synchronize raw material procurement, production of multiple items, and delivery to multiple retailers. Siajadi et al. [38] introduced a new methodology to obtain the joint economic lot size for a single-vendor multi-retailer problem. Chan and Kingsman [8] proposed a coordinated single-vendor multi-buyer supply chain model by synchronizing delivery and production cycles. Thangam and Uthayakumar [42] designed an approximate cost function through which a single-supplier and multi-retailer supply chain can identify optimal reorder points. Chen and Chang [9] aimed to jointly determine the optimal retail price, replenishment cycle, and shipment number for deteriorating items in a one-manufacturer and multi-retailer channel. Hoque [20, 21, 22] studied several synchronization policies between a single vendor and multiple buyers. Jha and Shanker [28] presented an integrated production-inventory model formulated to minimize the joint total expected cost.


Battini et al. [5] dealt with a multi-echelon inventory system in which one vendor supplies an item to multiple buyers, and developed a consignment stock inventory model in which many clients can establish a consignment stock policy with the same vendor. Ben-Daya et al. [7] modeled a consignment and vendor-managed inventory policy for a single-vendor, multi-buyer supply chain with known demand and studied three vendor-buyer partnerships. Chen et al. [13] dealt with the problem of coordinating a vertically separated distribution system under vendor-managed inventory and consignment arrangements with one wholesaler and multiple non-identical retailers. Adida and Ratisoontorn [1] investigated how competition among retailers influences supply chain decisions and profits under different consignment arrangements. Sarker [37] studied consignment stocking policy models for supply chain systems and comprehensively surveyed these models and performed associated critical analyses. Hariga and Al-Almari [18] designed integrated retail shelf space allocation and inventory models for a supply chain
operating under a vendor-managed inventory and consignment stock agreement. Ma-
teen et al. [33] discussed the interaction involving replenishment cycle between a vendor
and multiple retailers in a VMI system under stochastic demand. Glock and Kim [16]
studied a single-vendor-multi-retailer supply chain and considered the case where the
vendor merges with one of its retailers, and indicated that the type of competition is
of crucial importance for the structure of the supply chain after merging. However, as
we realized, these foregoing studies did not simultaneously address the inventory and
allocation decisions problem for vendors.

Another important issue is the variability of market demand. Historically it was
commonly accepted that random demand frequently occurs in competitive commodities
like perishable items. It is especially true for perishable items because of a short life cy-
CLE. Recent comparable studies on the newsvendor-type inventory model were interested
in the probabilistic random demand. Furthermore, most probabilistic demand-related
studies adopt independent normal demand for individual time periods. As we realized,
Bagchi and Hayya [2], Bagchi et al. [3], Silver et al. [39], Mantrala and Raman [32], Tang
et al. [41], McCardle et al. [34], Chen and Chen [10, 11], Desmet et al. [15], and Jha and
Shanker [28] assumed normally distributed demand in their analytical models. Using
a normal distribution to model demand on a given product appears to be convincing
because market demand is regularly an aggregate of numerous individual demand and
thus will most likely approximate a normal demand.

Nevertheless, it is questionable and unsuitable that normal distribution is taken as
a proxy for demand distribution as a result of the impossibility of negative demand.
Bartezzaghi et al. [4] argued that a probability distribution should be sought to imitate
demand distribution if its field is defined only for non-negative values. Among others,
the lognormal distribution is selected and considered as a more acceptable and valid
alternative to the normal distribution. The reason is that a variable with a lognormal
distribution only takes a value between zero and infinity and results in a normal distrib-
ution following the logarithmic operation, which is relatively easy to manipulate math-
ematically. Consequently, assuming that market demand for perishable items follows a
lognormal distribution should be theoretically justifiable and sustainable. Benavides et
al. [6], Huang et al. [25], and Huang [23, 24] also supported the lognormal distribution,
and used it to analyze and address the problems of demand forecasting. Presumably
more studies will make use of the lognormal distribution to model the demand variable
in the future.

In sum, this study considers the situation of a vendor responsible for producing and
delivering a quantity that matches expected aggregate demand from multiple retailers
during an upcoming selling period for a given perishable item and allocates this perish-
able inventory among retailers. For developing the optimal inventory-allocation decision
for vendor under above circumstance, this study seeks to extend the newsvendor model
to create the integrated inventory-allocation decision model for a given perishable item
associated with a number of independent lognormal demands from retailers. Notably,
besides lognormal demand distribution, the proposed model also incorporates Ito process
to capture demand manner and a comprehensive integrated approach for deducing the
aggregate demand. Finally, an effective and practical analytic method is developed to assist vendors of perishable items determine the inventory quantity and allocation decision that can maximize the expected profits. Through numerical examples, this study demonstrates that the optimal inventory-allocation solution can be straightforwardly obtained using the proposed inventory-allocation integrated model. Furthermore, a finding via sensitivity analysis is that the expected profits increase with decreased volatility of aggregate demand that can be realized by means of encouraging retail competition to reduce demand correlation between retailers.

2. Analytical Model Development

The purpose of this section is to develop a more plausible, effective and practicable vendor-managed inventory-allocation integrated model, which can maximize vendor’s profit, for given perishable item with stochastic demand in a single-vendor multi-retailer supply chain.

2.1. Modeling demand forecast

In contrast with the previous studies, the distinguishable characteristics in modeling demand forecast is as follows: (1) market demand for considered perishable items is assumed to comply with the lognormal distribution. (2) Ito process, which is usually used to model the behavior of financial assets price, is applied to suitably capture the demand shift by using the corresponding continuous-time differential equation. This study is confident that these two characteristics are better off in imitating and modeling the variability of real-life demand for a given perishable item. The Ito process is a stochastic process that possesses the Markov property and involves with a permanent component of regular trend and a temporary component of random diffusion. Let $D_t$ represent demand quantity during period $t$ for a given perishable item. The Ito process for random variable $D_t$ can then be represented algebraically as follows:

$$dD_t = \mu(D_t, t)dt + \sigma(D_t, t)dz_t.$$  \hspace{1cm} (2.1)

In the stochastic diffusion equation of Eq. (2.1), $\mu(D_t, t)$ represents the regular trend component, while $\sigma(D_t, t)$ represents the random diffusion component. Additionally, variable $z_t$ is assumed to satisfy the standard Wiener process (or say, Brownian motion), and its increment $dz_t = z_t - z_{t-1} = \varepsilon_t \sqrt{\tau}$, where $\varepsilon_t$ is a standard normal variable; that is, $\varepsilon_t \sim \phi(0, 1)$ and $d\tau$ represents a given small time interval, accordingly implying that $dz_t$ is a normal variable; namely $dz_t \sim \phi(0, \tau)$.

A special form $\mu(D_t, t) = \mu D_t$ and $\sigma(D_t, t) = \sigma D_t$ can be obtained when the diffusion component is roughly stationary (i.e., independent of time) that is expected to hold for the demand process of a typical perishable item over a relatively finite selling horizon. Where, parameters $\mu$ and $\sigma$ represent the expected rate of demand growth and the standard deviation of the rate of demand growth, respectively, and both are constant for all time periods. Eq. (2.1), thus, can be reformulated as follows:

$$dD_t = \mu D_t dt + \sigma D_t dz_t.$$  \hspace{1cm} (2.2)
Assume that $D_t$ is a lognormally distributed variable and let $f = \ln D_t$. Taylor expansion is applied to $f$ with truncating all but first two terms and then incorporated to Eq. (2.2) to yield the following discrete-time model:

$$D_t = D_0 \exp \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma \varepsilon \sqrt{t} \right], \quad (2.3)$$

where

- $D_0 =$ demand quantity during previous selling period,
- $t =$ length of time period,
- $\mu =$ expected annual rate of demand growth,
- $\sigma =$ standard deviation of rate of demand growth,
- $\varepsilon =$ a standard normal random variable; that is, $\varepsilon \sim \phi(0, 1)$.

Eq. (2.3) explicitly indicates that the expected value and variance of $\ln D_t$ are $E[\ln D_t] = \ln D_0 + (\mu - \sigma^2/2)$ and $\text{Var} [\ln D_t] = \sigma^2 t$, respectively.

### 2.2. Optimal inventory-allocation integrated model

The symbols used in the analytical model are defined as follows.

- $m =$ number of retailers,
- $T =$ length of selling period for a given perishable item (unit: years),
- $c =$ unit production/purchase cost (wholesale price) for a given perishable item,
- $p =$ unit selling price (retail price) for a given perishable item, which is equal for all retailers,
- $v =$ unit commission paid by the vendor to retailers, which is equal for all retailers,
- $h =$ unit holding cost for a given perishable item per period,
- $s =$ unit salvage value for a given perishable item ($s < c < p$),
- $r =$ unit shortage cost for a given perishable item ($r \geq p - c$, shortage cost would equal to opportunity cost (selling profit) plus potential goodwill loss),
- $D_i =$ demand quantity from retailer $i$ during the upcoming selling period for a given perishable item,
- $D_{0,i} =$ actual demand from retailer $i$ during the previous selling period for a given perishable item,
- $b_i =$ pre-unit adjustment cost from retailer $i$ for a given perishable item,
- $\mu_i =$ expected annual demand growth rate from retailer $i$ for a given perishable item,
- $\sigma_i =$ standard deviation of demand growth rate from retailer $i$ for a given perishable item, implying demand volatility for retailer $i$,
- $\sigma_{ij} =$ covariance of demand between retailers $i$ and $j$ for a given perishable item, implying demand correlation between retailers $i$, and $j$.

This study first assumes that the vendor has sufficient capacity to satisfy all retailer
demand. Also, quantity adjustment whereby retailers with excessive inventory transfer part inventory to those retailers who turn up a shortage is permitted and promoted. After doing so, the inventory quantity of perishable items in the supply chain can be utilized more efficiently to cut down unsold units and goodwill damage. As mentioned above, there are two decisions having to be adequately made for vendor in order to boost profitability. On the one hand, the inventory quantity approximates as closely as possible the aggregate demand during the upcoming selling period. On the other hand, inventory allocation among retailers match as accurately as possible to the demand of each retailer so as to lessen adjustment cost.

Let \( Q_i \) denote inventory-allocation quantity for retailer \( i \) during the upcoming selling period, which is the decision variable in this study. In this study, vendor profit is set equaling to subtract adjustment cost from selling gross profit. Expressed formally, the net profit function during the upcoming selling period for vendor depends on the aggregate demand and adjustment cost, and can be formulated as follows:

\[
R = \begin{cases} 
(p - c - v - h) \sum_{i=1}^{m} D_i - (c + h - s) \left( \sum_{i=1}^{m} Q_i - \sum_{i=1}^{m} D_i \right) \\
- \left[ \sum_{i=1}^{m} b_i \max(Q_i - D_i, 0) \right]; & 0 \leq \sum_{i=1}^{m} D_i \leq \sum_{i=1}^{m} Q_i \\
(p - c - v - h) \sum_{i=1}^{m} Q_i - r \left( \sum_{i=1}^{m} D_i - \sum_{i=1}^{m} Q_i \right) \\
- \left[ \sum_{i=1}^{m} b_i \max(Q_i - D_i, 0) \right]; & \sum_{i=1}^{m} D_i > \sum_{i=1}^{m} Q_i 
\end{cases}
\]

or, say

\[
R = \begin{cases} 
(p - s - v) D_S - (c + h - s) Q_S - \left[ \sum_{i=1}^{m} b_i \max(Q_i - D_i, 0) \right]; & 0 \leq D_S \leq Q_S \\
(p + r - c - v - h) Q_S - r D_S - \left[ \sum_{i=1}^{m} b_i \max(Q_i - D_i, 0) \right]; & D_S > Q_S 
\end{cases}
\]

(2.4)

Where the aggregate demand \( D_S = \sum_{i=1}^{m} D_i \) and aggregated inventory quantity \( Q_S = \sum_{i=1}^{m} Q_i \). It should be highlighted that the following novel integrated approach for the aggregate demand is another noticeable device and is confident of being contributive and referable to the future researches. Because the sum of a set of lognormal variables is not a lognormal, the aggregate demand \( D_S \) has to be equivalently transformed with the purpose of solving the geometric mean, which will be explained later, as follows.

\[
D_S = \sum_{i=1}^{m} D_i = \sum_{i=1}^{m} \left( \frac{D_i}{E[D_i]} \times E[D_i] \right) \\
= \sum_{i=1}^{m} \left( \frac{E[D_i]}{\sum_{j=1}^{m} (E[D_j])} \times \frac{D_i}{E[D_i]} \right) \times \sum_{j=1}^{m} (E[D_j])
\]
\[
\sum_{j=1}^{m} E[D_j] \times \sum_{i=1}^{m} (w_i \times D_i^*)
\]

where
\[
w_i = \frac{E[D_i]}{\sum_{j=1}^{m} E[D_j]} \quad \text{and} \quad \sum_{i=1}^{m} w_i = 1,
\]
\[
D_i^* = \frac{D_i}{E[D_i]},
\]
\[
E[D_i] = D_{0,i} \times e^{\mu_i T}.
\]

The arithmetic average of a set of lognormally distributed variables is not lognormal too whereas the geometric average is lognormal. Consequently, the arithmetic average must be replaced by the geometric average as so to fit for Eq. (2.3) through the following approximate transformation.

\[
\sum_{i=1}^{m} (w_i \times D_i^*) \cong \prod_{i=1}^{m} (D_i^*)^{w_i} - E \left[ \prod_{i=1}^{m} (D_i^*)^{w_i} \right] + E \left[ \sum_{i=1}^{m} (w_i \times D_i^*) \right],
\]

where,
\[
\prod_{i=1}^{m} (D_i^*)^{w_i} = \prod_{i=1}^{m} \left( \frac{D_i}{E[D_i]} \right)^{w_i} = \prod_{i=1}^{m} \left( \frac{D_i}{D_{0,i}} \right)^{w_i} \times \prod_{i=1}^{m} (e^{-\mu_i T})^{w_i}
\]
\[
= \prod_{i=1}^{m} \left( \frac{D_i}{D_{0,i}} \right)^{w_i} \times \left( \sum_{i=1}^{m} (-w_i \mu_i T) \right),
\]
\[
E \left[ \sum_{i=1}^{m} (w_i \times D_i^*) \right] = \sum_{i=1}^{m} (w_i \times E[D_i^*]) = \sum_{i=1}^{m} \left( w_i \times E \left[ \frac{D_i}{E[D_i]} \right] \right) = \sum_{i=1}^{m} w_i = 1. \tag{2.8}
\]

Based on (2.7), it can be straightforwardly worked out that

\[
\ln \left( \prod_{i=1}^{m} (D_i^*)^{w_i} \right) = \sum_{i=1}^{m} w_i \times \ln \left( \frac{D_i}{D_{0,i}} \right) - \sum_{i=1}^{m} (w_i \mu_i T). \tag{2.9}
\]

Moreover, because \(\ln(D_i/D_{0,i}); \quad i = 1, 2, \ldots, m\) within Eq. (2.9) are all normal distributions; that is, conforming to \(\phi((\mu_i - \sigma_i^2/2)T, \sigma_i^2 T)\), it follows that \(w_i \times \ln(D_i/D_{0,i})\) conform to normal distributions of \(\phi(w_i \times (\mu_i - \sigma_i^2/2)T, w_i^2 \sigma_i^2 T)\); \(i = 1, 2, \ldots, m\) as well.

For the sake of simplicity, let \(X = \prod_{i=1}^{m} (D_i^*)^{w_i}\), and then the following two expressions can be derived out on the basis of the definition of mean and variance.

\[
E \left[ \ln \left( \prod_{i=1}^{m} (D_i^*)^{w_i} \right) \right] = E[\ln X] = \sum_{i=1}^{m} [w_i(\mu_i - \sigma_i^2/2)T] - \sum_{i=1}^{m} (w_i \mu_i T)
\]
\[
= \sum_{i=1}^{m} (-w_i \sigma_i^2/2)T = \mu_X T, \tag{2.10}
\]
Var \left[ \ln \left( \prod_{i=1}^{m} (D_i^*)^{w_i} \right) \right] = \Var [\ln X] = \left( \sum_{i=1}^{m} w_i^2 \times \sigma_i^2 + \sum_{i=1}^{m} \sum_{j \neq i} w_i \times w_j \times \sigma_{ij} \right) T \\
= \sigma_X^2 T, \quad (2.11)

or following a shifting operation, say
\[
\mu_X = \sum_{i=1}^{m} \left( -w_i \sigma_i^2 / 2 \right), \quad \sigma_X = \left( \sum_{i=1}^{m} w_i^2 \times \sigma_i^2 + \sum_{i=1}^{m} \sum_{j \neq i} w_i \times w_j \times \sigma_{ij} \right)^{1/2}.
\]

Thus it turns out that
\[
E \left[ \prod_{i=1}^{m} (D_i^*)^{w_i} \right] = E[X] = e^{\mu_X T + \sigma_X^2 T/2}. \quad (2.12)
\]

Eq. (2.6) can be rewritten according to Eq. (2.12) as follows.
\[
\sum_{i=1}^{m} (w_i \times D_i^*) \approx \prod_{i=1}^{m} (D_i^*)^{w_i} - E \left[ \prod_{i=1}^{m} (D_i^*)^{w_i} \right] + E \left[ \sum_{i=1}^{m} (w_i \times D_i^*) \right] = X - A + 1, \quad (2.13)
\]
where,
\[
A = E[X] = e^{\mu_X T + \sigma_X^2 T/2}.
\]

Letting \( B = \sum_{i=1}^{m} (E[D_i]) \), then in accordance with Eq. (2.5) the aggregate demand can be re-expressed as follows
\[
D_S = \sum_{i=1}^{m} (E[D_i]) \times \sum_{i=1}^{m} (w_i \times D_i^*) = B \times (X - A + 1). \quad (2.14)
\]

Following the above result substitutes for \( D_S \) in Eq. (2.4), the net profit function of vendor thus can be reformulated as follows.
\[
R = \begin{cases} 
(p-s-v)B \times X - (p-s-v)B(A-1) - (c+h-s)Q_S \\
- \left[ \sum_{i=1}^{m} b_i \max(Q_i-D_i, 0) \right]; & 0 \leq X \leq Q_S/B + A - 1 \\
[p+r-c-v-h)Q_S + rB(A-1)] - rB \times X \\
- \left[ \sum_{i=1}^{m} b_i \max(Q_i-D_i, 0) \right]; & X > Q_S/B + A - 1.
\end{cases} \quad (2.15)
\]

Accordingly, vendor expected profit is then deduced from Eq. (2.15) and can be expressed according to the definition of expected value as follows:
\[
E[R] = (p - s - v)B \int_{0}^{Q_S/B + A - 1} X f(X) dX
\]
\[ -[(p - s - v)B(A - 1) + (c + h - s)Q_S] \int_0^{Q_S/B + A - 1} f(X) dX + [(p + r - c - v - h)Q_S + rB(A - 1)] \int_{Q_S/B + A - 1}^{\infty} f(X) dX \]
\[ -rB \int_{Q_S/B + A - 1}^{\infty} X f(X) dX \]
\[ -\left\{ \sum_{i=1}^{m} b_i \left[ Q_i \int_0^{Q_i} f(D_i) dD_i - \int_0^{Q_i} D_i f(D_i) dD_i \right] + \left( \int_{Q_i}^{\infty} D_i f(D_i) dD_i - Q_i \int_{Q_i}^{\infty} f(D_i) dD_i \right) \right\}. \]

(2.16)

As stated previously, the demand \((D_i)\) from each retailer during a certain selling period for a given perishable item is assumed to follow a lognormal distribution, and thus the probability distribution for the geometric average \((X)\) of demand \((D_i)\) is also lognormal distribution. Thus, the probability density function of variables \(D_i\) and \(X\) can be respectively expressed as follows:

\[
f(D_i) = \frac{1}{D_i \sigma_{D_i} \sqrt{T} \sqrt{2\pi}} e^{-\frac{(\ln D_i - E[\ln D_i])^2}{2\sigma_{D_i}^2 T}},
\]
\[
f(X) = \frac{1}{X \sigma_X \sqrt{T} \sqrt{2\pi}} e^{-\frac{(\ln X - E[\ln X])^2}{2\sigma_X^2 T}}.
\]

**Theorem.** Expected profit of vendor is worked out as follows:

\[
E[R] = (p + r - s - v) \left\{ \left[ Q_S + AB - B \right] N(d_{01}) - AB N(d_{02}) \right\} + (p - s - v)B - (c + h - s)Q_S \]
\[
- \left\{ \sum_{i=1}^{m} b_i \left[ 2 \left( D_{0,i} e^{\mu_{D_i} T} N(d_{i2}) - Q_i N(d_{i1}) \right) + Q_i - D_{0,i} e^{\mu_{D_i} T} \right] \right\},
\]

(2.17)

where,

\[
d_{01} = \frac{\ln[B/(Q_S + AB - B)] + \mu_X T}{\sigma_X \sqrt{T}},
\]
\[
d_{02} = \frac{\ln[B/(Q_S + AB - B)] + (\mu_X + \sigma_X^2) T}{\sigma_X \sqrt{T}},
\]
\[
d_{i1} = \frac{\ln[D_{0,i}/Q_i] + (\mu_{D_i} - \sigma_{D_i}^2/2) T}{\sigma_{D_i} \sqrt{T}},
\]
\[
d_{i2} = \frac{\ln[D_{0,i}/Q_i] + (\mu_{D_i} + \sigma_{D_i}^2/2) T}{\sigma_{D_i} \sqrt{T}}.
\]

Where, function \(N(x)\) is the cumulative distribution function for a standardized normal variable. In other words, it is the probability that a variable with a standard normal distribution, \(\phi[0,1]\), will be less than \(x\).
Proof. The deduction is relegated to Appendix.

Taking the first derivative of $E[R]$ with respect to order quantity $Q_i; i = 1, 2, \ldots, m$ reveals that

$$
\frac{\partial E[R]}{\partial Q_i} = (p + r - s - v)N(d(0)) - \frac{n(d(0))}{\sigma_X \sqrt{T}} + AB \frac{n(d(2))}{\sigma_X \sqrt{T}(Q_S + AB - B)} - (c + h - s) - b_i [2 \left( - D_0, e^{\mu_i T} \frac{n(d(2))}{\sigma_D \sqrt{T} Q_i} - N(d(1)) + \frac{n(d(1))}{\sigma_D \sqrt{T}} \right) + 1]; \ i = 1, 2, \ldots, m. \ (2.18)
$$

Where $n(d(0)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}$, $n(d(2)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}$, $n(d(1)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}$, and $n(d(2)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}$. Moreover, $n(d(2))$ can be substituted for $n(d(2))$ via the following transformation

$$
n(d(0)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2} - \sigma_X \sqrt{T}^2/2} = \frac{1}{\sqrt{2\pi}} \left( e^{-\frac{d^2}{2} + \sigma_X \sqrt{T}^2} e^{-\sigma_X^2 T/2} \right) = \frac{1}{\sqrt{2\pi}} \left( e^{-\frac{d^2}{2} + \ln(B/(Q_S + AB - B)) + (\mu_X + \sigma_X^2 T/2) T} \right) = AB \times \frac{n(d(2))}{Q_S + AB - B}.
$$

Likewise, through similar logic $n(d(1))$ can be substituted with $n(d(2))$ via the following transformation

$$
n(d(1)) = D_0, e^{\mu_i T} \times \frac{n(d(2))}{Q_i}.
$$

Consequently, it turns out that

$$
\frac{\partial E[R]}{\partial Q_i} = (p + r - s - v)N(d(0)) - (c + h - s) - b_i [1 - 2N(d(1))]; \ i = 1, 2, \ldots, m. \ (2.19)
$$

The maximal value of $E[R^*]$ occurs at $Q_i^*$, satisfying jointly $\partial E[R]/\partial Q_i = 0; i = 1, 2, \ldots, m$. Therefore, the optimal inventory-allocation decision $Q_i^*; i = 1, 2, \ldots, m$ are determined first through simultaneously resolving $\partial E[R]/\partial Q_i = 0; i = 1, 2, \ldots, m$. Although a closed-form expression cannot be obtained, some available numerical software can facilitate to find out the solution. Subsequently, the maximal expected profit $E[R^*]$ can be estimated using Eq. (2.17), immediately after determining $Q_i^*; i = 1, 2, \ldots, m$. Finally, this study takes a supply chain system composed of two retailers $i$ and $j$ as an illustration to prove the concavity of proposed analytic model. Since

$$
U = \frac{\partial^2 E[R]}{\partial Q_i^2} = -\left[ (p + r - s - v) \frac{n(d(0))}{\sigma_X \sqrt{T}(Q_S + AB - B)} + 2b_i \frac{n(d(1))}{\sigma_D \sqrt{T} Q_i} \right] < 0,
$$

$$
V = \frac{\partial^2 E[R]}{\partial Q_i \partial Q_j} = -\left[ (p + r - s - v) \frac{n(d(0))}{\sigma_X \sqrt{T}(Q_S + AB - B)} \right],
$$
\begin{equation}
W = \frac{\partial^2 E[R]}{\partial Q_j^2} = -\left[ \frac{(p + r - s - v)n(d_{01})}{\sigma_X \sqrt{T_Q S + AB - B}} + 2b_j \frac{n(d_{j1})}{\sigma_{D_j} \sqrt{T_Q j}} \right],
\end{equation}

\begin{equation}
Y = V^2 - UW = -\left[ \frac{(p + r - s - v)n(d_{01})}{\sigma_X \sqrt{T_Q S + AB - B}} \times 2 \left( \frac{b_in(d_{i1})}{\sigma_{D_i} \sqrt{T_Q i}} + \frac{b_jn(d_{j1})}{\sigma_{D_j} \sqrt{T_Q j}} \right) + 4 \frac{b_ib_jn(d_{i1})n(d_{j1})}{\sigma_{D_i} \sigma_{D_j} T_Q i T_Q j} \right] < 0,
\end{equation}

the existence of maximal value can thus be verified.

### 2.3. Model parameters estimation

The expected growth rate of demand \( \mu_i \), standard deviation of growth rate \( \sigma_i \), and covariance \( \sigma_{ij} \) can be determined from the sample estimates \( \hat{\mu}_i, \hat{\sigma}_i \) and \( \hat{\sigma}_{ij} \), respectively, based on historical retailer demand data. Assuming two samples of demand data for retailers \( i \) and \( j \) for past \( N \) time periods, namely, \( D_{i1}, D_{i2}, \ldots, D_{iN} \) and \( D_{j1}, D_{j2}, \ldots, D_{jN} \), each with length \( \Delta \tau \), for a given perishable item, the logarithmic growth rate of demand \( r_t \) for the demand time series during period \( t \) is as follows:

\begin{equation}
r_{kt} = \ln \left( \frac{D_{kt}}{D_{kt-1}} \right); \quad t = 2, 3, \ldots, N, \quad k = i, j.
\end{equation}

Additionally, for retailer \( k \) (\( k = i, j \)) and during period \( t \), the logarithmic growth rate of demands \( r_{kt}, t = 2, 3, \ldots, N \) all share an independent and identical normal distribution with mean \( \bar{r}_k \) and standard deviation \( s_k \).

Accordingly, for retailer \( k; k = i, j \), and the three estimates \( \hat{\mu}_k, \hat{\sigma}_k \) and \( \hat{\sigma}_{ij} \) can be calculated as follows:

\begin{equation}
\hat{\mu}_k = \bar{r}_k + \frac{s_k^2}{2\Delta \tau}, \quad \text{and} \quad \hat{\sigma}_k = \frac{s_k}{\sqrt{\Delta \tau}},
\end{equation}

\begin{equation}
\bar{r}_k = \frac{\sum_{t=2}^{N} r_{kt}}{N - 1}, \quad \text{and} \quad s_k = \sqrt{\frac{\sum_{t=2}^{N} (r_{kt} - \bar{r}_k)^2}{N - 2}},
\end{equation}

\begin{equation}
\hat{\sigma}_{ij} = \frac{\sum_{t=2}^{N} (r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)}{N - 2}.
\end{equation}

### 3. Numerical Experiment

This section presents a numerical example involving a supply chain that consists of one vendor and five retailers, and demonstrates that the proposed analytical model can optimize inventory quantity and inventory-allocation decisions to maximize the expected profit for vendor. Meanwhile, the model parameters required in the numerical example are arranged for a plausible perishable item, yielding a corresponding value set comprising \( (T, p, c, v, h, s, r) = (0.5, $100, $60, $15, $2, $10, $150) \). Additionally, the following
demand quantities during the previous selling period, expected annual demand growth rates, adjustment cost per unit, and variance and covariance of demand growth rates for those five retailers are hypothetically acquired from historical demand data based on Section 2.3 and used in this example:

\[
D_0 = \begin{bmatrix}
10,000 \\
15,000 \\
30,000 \\
8,000 \\
50,000 \\
\end{bmatrix}, \quad \mu = \begin{bmatrix}
0.15 \\
0.2 \\
0.5 \\
-0.1 \\
0.3 \\
\end{bmatrix}, \quad b = \begin{bmatrix}
$2 \\
$5 \\
$1 \\
$8 \\
$3 \\
\end{bmatrix}, \quad \sigma = \begin{bmatrix}
0.0400 & 0.0420 & -0.0100 & 0.0120 & -0.0300 \\
0.0420 & 0.1225 & 0.0263 & -0.0735 & 0.0750 \\
-0.0100 & 0.0263 & 0.0625 & 0.0750 & 0.0188 \\
0.0120 & 0.0735 & -0.0750 & 0.3600 & 0.1350 \\
-0.0300 & 0.0750 & 0.0188 & 0.1350 & 0.2500 \\
\end{bmatrix}.
\]

By using the numerical program offered by MATLAB software, the optimal inventory-allocation decision and relative percentage for every retailer \(i, i = 1, 2, \ldots, 5\) for the above parameter settings using the proposed analytical method and numerical solution procedure are readily solved as

\[
Q^* = \begin{bmatrix}
11,065 \\
16,486 \\
41,647 \\
7,144 \\
57,942 \\
\end{bmatrix}, \quad \text{and} \quad \gamma_{Q^*} = \begin{bmatrix}
0.0824 \\
0.1228 \\
0.3101 \\
0.0532 \\
0.4315 \\
\end{bmatrix}.
\]

Meanwhile, the optimal aggregated inventory quantity and expected profit for vendor are calculated as \(Q^*_S = 134,283\) and \(E[R^*] = $1,636,950\), respectively.

To examine concavity this study presents two scenarios to compare the expected profits. The first scenario designates a given range of aggregated inventory quantities, which is allocated among five retailers based on the optimal relative percentage \(\gamma Q^*_S\). The second scenario involves the aggregated inventory quantity remaining the same as \(Q^*_S\) for all cases and inventory-allocation decisions for these five retailers are sequentially adjusted in pairs, where each pair comprises one variable that increases by a certain percentage and another that decreases by an identical percentage relative to the identified optimal relative percentage \(\gamma Q^*_S\).

While Table 1 lists the corresponding expected profits for the first scenario, in which aggregated inventory quantities range from 90,000 to 180,000 in increments of 2,500, and Table 2 lists numerical results for the second scenario, in which the inventory-allocation quantity of each retailer is adjusted in turn by percentages of 0.1%, 0.5% and 1%. Based on the results listed in Table 1, Figure 1 illustrates the variability of expected profit, clearly demonstrating that it first increases and then decreases with rising inventory quantities, ultimately reaching (as expected) a maximum of $1,636,950 for an inventory quantity of 134,283, thus validating the solution and confirming the concavity.

Besides, Figure 1 illustrates that expected profits for smaller inventory quantities tend to be lower than those for larger inventory quantities owing to the shortage cost considerably exceeding the salvage value. On the other hand, the examination of expected profit shown in Table 2 also clearly reveals that, as expected, none of the fine-tuned
Table 1: Comparison of expected profits for a given range of inventory quantities.

<table>
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<tr>
<th>Qs</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>E[R]</th>
</tr>
</thead>
<tbody>
<tr>
<td>90,000</td>
<td>7,416</td>
<td>11,049</td>
<td>27,913</td>
<td>4,788</td>
<td>38,834</td>
<td>-172,583</td>
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<td>11,356</td>
<td>28,688</td>
<td>4,921</td>
<td>39,913</td>
<td>2,403</td>
</tr>
<tr>
<td>95,000</td>
<td>7,828</td>
<td>11,663</td>
<td>29,463</td>
<td>5,054</td>
<td>40,992</td>
<td>172,943</td>
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<tr>
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<td>8,034</td>
<td>11,970</td>
<td>30,239</td>
<td>5,187</td>
<td>42,070</td>
<td>338,140</td>
</tr>
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<td>8,240</td>
<td>12,277</td>
<td>31,014</td>
<td>5,320</td>
<td>43,149</td>
<td>497,055</td>
</tr>
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<td>102,500</td>
<td>8,446</td>
<td>12,584</td>
<td>31,789</td>
<td>5,453</td>
<td>44,228</td>
<td>648,724</td>
</tr>
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<td>8,652</td>
<td>12,891</td>
<td>32,565</td>
<td>5,586</td>
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<td>38,767</td>
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<td>1,633,927</td>
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</table>

In conclusion, the concavity of the proposed analytical model can be identified by reason of evidences from two experimental scenarios.
Table 2: Comparison of expected profits for various inventory-allocation decisions under three fine-tuning rates.

<table>
<thead>
<tr>
<th>Q&lt;sub&gt;1&lt;/sub&gt;</th>
<th>Q&lt;sub&gt;2&lt;/sub&gt;</th>
<th>Q&lt;sub&gt;3&lt;/sub&gt;</th>
<th>Q&lt;sub&gt;4&lt;/sub&gt;</th>
<th>Q&lt;sub&gt;5&lt;/sub&gt;</th>
<th>E[R]</th>
</tr>
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<tr>
<td><strong>Optimal</strong></td>
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<td>16,486</td>
<td>41,647</td>
<td>7,144</td>
<td>57,942</td>
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<tr>
<td>0.1%</td>
<td>11,199</td>
<td>16,486</td>
<td>41,512</td>
<td>7,144</td>
<td>57,942</td>
</tr>
<tr>
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<td>11,199</td>
<td>16,486</td>
<td>41,512</td>
<td>7,144</td>
<td>57,942</td>
</tr>
<tr>
<td></td>
<td>11,199</td>
<td>16,486</td>
<td>41,512</td>
<td>7,144</td>
<td>57,942</td>
</tr>
<tr>
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<td>41,647</td>
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</tr>
<tr>
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<td>17,157</td>
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<td>10,393</td>
<td>17,157</td>
<td>42,318</td>
<td>7,144</td>
<td>57,942</td>
</tr>
<tr>
<td>1%</td>
<td>12,407</td>
<td>16,486</td>
<td>42,989</td>
<td>7,144</td>
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<td>12,407</td>
<td>16,486</td>
<td>42,989</td>
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<tr>
<td></td>
<td>9,722</td>
<td>16,486</td>
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<td>9,722</td>
<td>16,486</td>
<td>42,989</td>
<td>7,144</td>
<td>57,942</td>
</tr>
</tbody>
</table>

\( D_{n,i}^T \) = 11,883  
\( D_{n,i}^{T,T} \) = 11,000  
16,918  
39,312  
7,666  
59,286  
1,636,089
Additionally, this study performs sensitivity analysis for key parameters. Among the model parameters, demand volatility, which is commonly measured in terms of the standard deviation of demand growth rate, indicates uncertain future market demand and is crucial in the inventory-allocation decision. As generally realized, since demand volatility often negatively impacts expected profits, in practice vendors wish to reduce demand volatility in order for improving their profitability. On the other hand, volatility of aggregate demand potentially projects the competitive intensity for a given perishable item, and in accordance with Eq. (2.11) this competitive intensity can be diminished by three approaches, namely increasing the number of retailers, decreasing demand volatility of individual retailer, and reducing demand covariance between retailers. Among others, one practical tactic for reducing aggregate demand volatility is to decrease demand covariance between retailers by encouraging retail competition and had better attain a negative demand correlation.

More specifically, aggregate demand volatility is not simply a weighted average of individual demand volatility except that overall pairs of retailer exhibit perfect positive correlation between demands each other. Aggregate demand volatility also considers how the demands of multiple retailers are subject to co-variance such that aggregate demand volatility for a group of retailers is generally smaller than the weighted average of individual demand volatility. This is the advantage of diversification. In this numerical instance, for example, the volatility of aggregate demand is 0.2875, while the simple weighted average of individual demand volatility is considerably higher at 0.3891, suggesting that expected profit only reaches $1,145,356 and lowers 30% under this volatility level. To conclude, increases in number of retailers and decreases in demand correlation between retailers are two practicable and effective strategies that can ease aggregate demand volatility for vendors.

To better understand the volatility of aggregate demand, this study performs detailed sensitivity analysis against demand volatility and sets the value of the standard deviation
(volatility) of aggregate demand ranging from 5% to 95%, within which range it changes in 5% increments, to determine the corresponding optimal inventory quantity, inventory-allocation decision, and maximal expected profit for vendor. Table 3 lists the numerical results for the specified range of volatilities of aggregate demand. The Table obviously shows that the maximal expected profit gradually declines with increasing volatility, and even becomes negative when volatility approaches 65%. Definitely, there is no sense doing business under such circumstances, but it is unusual for demand volatility to be so high in practice. On average, a 5% increment in volatility incurs a loss of 243,647 (or say 24.9311%) in maximal expected profit. Unsurprisingly, the volatility negatively impacts vendor profit.

Additionally, for reasons of perception, Figure 2 also shows the results of sensitivity analysis originated from volatile and uncertain aggregate demand. The profile also reveals a clear divergence between optimal inventory quantity and expected profit in response to changes in demand volatility. The optimal inventory quantity initially increases with volatility, meaning vendors should extend production/order to prevent expensive losses associated with shortage units in circumstances of increased aggregate demand volatility. Remarkably, a transition point appears in the region of 30% of volatility, and the optimal inventory quantity subsequently decreases with volatility, suggesting that in the present circumstances, the potential loss incurred from unsold units exceeds the shortage cost. Additionally, Figure 2 also reveals that the increase in volatility of aggregate demand continuously and negatively influences expected profit. Being consistent with above observations, high demand volatility generally negatively impacts vendor profitability, and in this condition expanding numbers of new retailers and intensifying competition among retailers to reduce the covariance between demands can help decrease aggregate demand volatility and thus increase expected profit.

Figure 2: Effect of changes in volatility of aggregate demand on optimal inventory quantity and maximal expected profit.
Another manageable crucial parameter is adjustment cost and thus this study also observes and analyzes its sensitivity. To this end, assuming the remaining parameters are constant, adjustment cost of each retailer varies in sequence with a percentage ranging from -50% to 50%, and changing in increments of 20%. Table 4 lists the variant effects of adjustment cost changes for every retailer on the optimal inventory quantity, inventory-allocation decision, and expected profit. Also, the bold numbers in Table 4 indicate the consequences when adjustment costs for all five retailers simultaneously change at the same rate. To compare the differences in effects among retailers, Figure 3, which is drawn from Table 4, further illustrates the separate results for variability of expected profit of each retailer.

Figure 3 clearly shows that retailers 5 and 4 have respectively the most significant and second most significant effects of adjustment cost on vendor expected profit. Consequently, the profitability could be substantially improved if adjustment costs for retailers 5 and/or 4 are reduced. Additionally, Table 4 also reveals that, as generally expected, both optimal inventory quantity and maximal expected profit constantly and negatively vary with overall adjustment cost, and on average optimal inventory quantity only slightly decreases, by about 0.0318%, while maximal expected profit comparably declines, by about 1.1001% (or say $18,122), given an increase of 20% (or say $1.1) in overall adjustment cost. It can thus be seen that a reduction in adjusting frequency and adjustment cost, especially for more sensitive retailers, also is quite a relevant element to increase profitability.
Table 4: Resulting optimal inventory quantities and maximal expected profits for given variation rates of adjustment cost. Variation rate.

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<th>( Q^*_1 )</th>
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<th>( Q^*_3 )</th>
<th>( Q^*_4 )</th>
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4. Concluding Remarks

This study extends a typical newsvendor model to incorporate vendor inventory-allocation integrated decision in a single-vendor multi-retailer supply chain system, which has been becoming a prevalent channel arrangement in practice. This study endeavors to develop the inventory-allocation integrated decision model from a vendor viewpoint.
Therefore, the decision model developed here is applicable to widespread vendor-managed inventory supply chain systems. Moreover, this decision model should be particularly valuable and necessary for perishable items because negligible salvage value often remains on unsold units, excessive shortage costs for stock-out units are punished, and a substantial adjustment cost is frequently incurred for frequent adjustments between retailers due to imprecise allocation. By contract, this study assumes that individual retailer demand for a given perishable item during a selling period exhibits a lognormal distribution, which is considered more reasonable than the familiar normal distribution. Additionally, it is special and notable in this study that Ito process is applied to imitate and model the stochastic shift behavior of market demand and a novel and comprehensive geometric average transformation device is employed to address the problem of non-lognormal aggregate demand.

After some efforts, this study finally develops an effectual and practicable analytical model for optimizing inventory quantity and inventory-allocation decision so as to maximize vendor expected profit during the upcoming selling period. This study takes a plausibly supposed numerical instance to demonstrate that the proposed analytical model can, as expected, solve the optimal inventory quantity and inventory-allocation decision for vendor to earn the maximal expected profit. Additionally, sensitivity analysis is performed for the crucial parameter volatility of aggregate demand and adjustment cost, and reveals some interesting and noticeable managerial insights. In summary, the analytical model presented herein and the experimental findings can help vendors, who trade in perishable items in the case of single-vendor multi-retailer supply chain systems, improve their profitability. The applicable future researches based on the works in this study include, for example, incorporating wholesale and/or retail pricing policies, dual channels composed of direct channel and retailer channel, synchronizing joint

Figure 3: Effect of changes in individual retailer adjustment cost on maximal expected profit.
replenishment and delivery cycles, allowing for returns policies, involving multiple perishable items, and extending into a multi-vendor multi-retailer supply chain with vendors competition.

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Appendix. Proof of Expected Vendor Profit

Eq. (2.16) can be solved by first solving the four underlying components embraced in the equation, as detailed below, then linking these four components to return to the original expression.

1. \[ \int_{Q_S/B+A-1}^{\infty} f(X) dX + \int_{Q_i}^{\infty} f(D_i) dD_i. \]

This component can be algebraically derived as follows:

\[ \int_{Q_S/B+A-1}^{\infty} f(X) dX = \int_{Q_S/B+A-1}^{\infty} \frac{1}{X \sigma_X \sqrt{2\pi}} e^{-\left(\frac{(\ln X - E[\ln X])^2}{2\sigma_X^2}\right)} dX. \] (A.1)

Let \( \ln X = s, E(\ln X) = \bar{s} \) and \( \sigma_X \sqrt{T} = u \); Eq. (A.1) can then be transformed into the following expression.

\[ \int_{Q_S/B+A-1}^{\infty} \frac{1}{X u \sqrt{2\pi}} e^{-\left(\frac{(s-\bar{s})^2}{2u^2}\right)} dX. \] (A.2)

Furthermore, if \( w = \frac{(s-\bar{s})}{u} \), then \( dw = \frac{1}{uX} dX \) through differentiation. The lower bound of the integral for \( w \) is accordingly transformed as follows:

\[ = \ln\left(\frac{(Q_S + AB - B)/B}{\ln X}\right) - \frac{\ln(Q_S + AB - B) - \ln B - \mu_X T}{\sigma_X \sqrt{T}} \]

\[ = \ln\left(\frac{B/(Q_S + AB - B)}{\ln X}\right) + \frac{\mu_X T}{\sigma_X \sqrt{T}} = -d_0. \]

Again, after applying \( dw \) and the lower bound of the integral for \( w \) in Eq. (A.2), the equation can be reformulated as

\[ \int_{Q_S/B+A-1}^{\infty} \frac{1}{X u \sqrt{2\pi}} e^{-\left(\frac{(s-\bar{s})^2}{2u^2}\right)} dX = \int_{-d_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{w^2}{2}\right)} dw = N(d_0). \] (A.3)

Similarly, the following expression can also be obtained through the above procedure.

\[ \int_{Q_i}^{\infty} f(D_i) dD_i = N(d_{i1}), \] (A.4)
\[d_{i1} = \frac{\ln[D_{0,i}/Q_i] + (\mu_{D_i} - \sigma^2_{D_i}/2)T}{\sigma_{D_i}\sqrt{T}}.\]

(2). \[\int_{Q_i}^{\infty} X f(X)dX\] and \[\int_{Q_i}^{\infty} D_i f(D_i)dD_i.\]

The component can be deduced in a similar way. First, the component can be expanded as:

\[\int_{Q_i}^{\infty} X f(X)dX = \int_{Q_i}^{\infty} \frac{1}{\sigma_X \sqrt{T}} e^{-\frac{(\ln X - E[\ln X])^2}{2\sigma^2_X}} dX.\]  
(A.5)

The integral can be obtained via the same procedure used for component (1):

\[\int_{Q_i}^{\infty} \frac{1}{\sigma_X \sqrt{T}} e^{-\frac{(\ln X - E[\ln X])^2}{2\sigma^2_X}} dX = A e^{-\frac{\ln A}{\sigma_X \sqrt{T}}} \int_{Q_i}^{\infty} \frac{1}{u \sqrt{2\pi}} e^{-\frac{(y - \bar{y})^2}{2u^2}} dX.\]  
(A.6)

Let \(y = \frac{s - (\bar{s} + u^2)}{u}\), then \(dy = \frac{1}{u} ds = \frac{1}{u} d(\ln X) = \frac{1}{uX} dX\) can be obtained. Likewise, the lower bound of the integral for \(y\) is transformed as follows:

\[\frac{s - (\bar{s} + u^2)}{u} = \frac{\ln[(Q_i + AB - B)/B] - (E[\ln X] + \sigma^2_X T)}{\sigma_X \sqrt{T}}\]

\[= \frac{\ln[B/(Q_i + AB - B) - (\mu_X T + \sigma^2_X T)]}{\sigma_X \sqrt{T}} = \frac{\ln[B/(Q_i + AB - B)] + (\mu_X T + \sigma^2_X T)}{\sigma_X \sqrt{T}}\]

\[= -d_{02}.\]

Including \(dy\) and the lower bound of the integral for \(y\) in Eq. (A.6) yields the following equation:

\[\int_{Q_i}^{\infty} X f(X)dX = A \int_{-d_{02}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = A[1 - N(-d_{02})] = AN(d_{02}).\]  
(A.7)

Similarly, the deduction of \[\int_{Q_i}^{\infty} D_i f(D_i)dD_i\] results in the following.

\[\int_{Q_i}^{\infty} D_i f(D_i)dD_i = D_{0,i} e^{\mu_{D_i} T} N(d_{i2}),\]  
(A.8)

\[d_{i2} = \frac{\ln[D_{0,i}/Q_i] + (\mu_{D_i} + \sigma^2_{D_i}/2)T}{\sigma_{D_i}\sqrt{T}}.\]

(3). \[\int_{0}^{Q_i} f(X)dX\] and \[\int_{0}^{Q_i} f(D_i)dD_i\]
Because the deduction of this component closely resembles that for component (1),
the procedure is not detailed. The following two close-form formulas are also identified:

\[
\int_0^{Q_s/B+A-1} f(X)dX = N(-d_{01}) = 1 - N(d_{01}). \quad (A.9)
\]

\[
\int_0^{Q_i} f(D_i)dD_i = 1 - N(d_{i2}). \quad (A.10)
\]

Likewise, this component is deduced in much the same manner as component (2).
Thus, only the final outcomes are presented, as follows:

\[
\int_0^{Q_s/B+A-1} Xf(X)dX = AN(-d_{02}) = A[1 - N(d_{02})], \quad (A.11)
\]

\[
\int_0^{Q_i} D_if(D_i)dD_i = D_{0,i}e^{\mu_iT}[1 - N(d_{i2})]. \quad (A.12)
\]

By applying the results of Eqs. (A.3), (A.4), (A.7), (A.8), (A.9), (A.10), (A.11) and
(A.12) to Eq. (2.16), the expected vendor profit can thus be solved.

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